Lecture Notes for Economics 210: Macroeconomic Theory I*

Dirk Krueger
Stanford University

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1 Overview and Summary

After a quick warm-up for dynamic general equilibrium models in the first part of the course we will discuss the two workhorses of modern macroeconomics, the neoclassical growth model with infinitely lived consumers and the Overlapping Generations (OLG) model. This first part will focus on techniques rather than issues; one first has to learn a language before composing poems.

I will first present a simple dynamic pure exchange economy with two infinitely lived consumers engaging in intertemporal trade. In this model the connection between competitive equilibria and Pareto optimal equilibria can be easily demonstrated. Furthermore it will be demonstrated how this connection can exploited to compute equilibria by solving a particular social planners problem, an approach developed first by Negishi (1960) and discussed nicely by Kehoe (1989).

This model with then enriched by production (and simplified by dropping one of the two agents), to give rise to the neoclassical growth model. This model will first be presented in discrete time to discuss discrete-time dynamic programming techniques; both theoretical as well as computational in nature. The main reference will be Stokey et al., chapters 2-4. As a first economic application the model will be enriched by technology shocks to develop the Real Business Cycle (RBC) theory of business cycles. Cooley and Prescott (1995) are a good reference for this application. In order to formulate the stochastic neoclassical growth model notation for dealing with uncertainty will be developed.

This discussion will motivate the two welfare theorems, which will then be presented for quite general economies in which the commodity space may be infinite-dimensional. We will draw on Stokey et al., chapter 15's discussion of Debreu (1954).

The next two topics are logical extensions of the preceding material. We will first discuss the OLG model, due to Samuelson (1958) and Diamond (1965). The first main focus in this module will be the theoretical results that distinguish the OLG model from the standard Arrow-Debreu model of general equilibrium: in the OLG model equilibria may not be Pareto optimal, fiat money may have positive value, for a given economy there may be a continuum of equilibria and the (core of the economy may be empty). All this could not happen in the standard Arrow-Debreu model. References that explain these differences in detail include Geanakoplos (1989) and Kehoe (1989). Our discussion of these issues will largely consist of examples. One reason to develop the OLG model was the uncomfortable assumption of infinitely lived agents in the standard neoclassical growth model. Barro (1974) demonstrated under which conditions (operative bequest motives) an OLG economy will be equivalent to an economy with infinitely lived consumers. One main contribution of Barro was to provide a formal justification for the assumption of infinite lives. As we will see this methodological contribution has profound consequences for the macroeconomic effects of government debt, reviving the Ricardian Equivalence proposition. As a prelude we will briefly discuss Diamond's (1965) analysis of government debt in an OLG model.

In the next module we will discuss the neoclassical growth model in continuous time to develop continuous time optimization techniques. After having learned the technique we will review the main developments in growth theory and see how the various growth models fare when being contrasted with the main empirical findings from the Summers-Heston panel data set. We will briefly discuss the Solow model and its empirical implications (using the article by Mankiw et al. (1992) and Romer, chapter 2), then continue with the Ramsey model (Intriligator, chapter 14 and 16, Blanchard and Fischer, chapter 2). In this model growth comes about by introducing exogenous technological progress. We will then review the main contributions of endogenous growth theory, first by discussing the early models based on externalities (Romer (1986), Lucas (1988)), then models that explicitly try to model technological progress (Romer (1990).

All the models discussed up to this point usually assumed that individuals are identical within each generation (or that markets are complete), so that without loss of generality we could assume a single representative consumer (within each generation). This obviously makes life easy, but abstracts from a lot of interesting questions involving distributional aspects of government policy. In the next section we will discuss a model that is capable of addressing these issues. There is a continuum of individuals. Individuals are ex-ante identical (have the same stochastic income process), but receive different income realizations ex post. These income shocks are assumed to be uninsurable (we therefore depart from the Arrow-Debreu world), but people are allowed to self-insure by borrowing and lending at a risk-free rate, subject to a borrowing limit. Deaton (1991) discusses the optimal consumption-saving decision of a single individual in this environment and Aiyagari (1994) incorporates Deaton's analysis into a full-blown dynamic general equilibrium model. The state variable for this economy turns out to be a cross-sectional distribution of wealth across individuals. This feature makes the model interesting as distributional aspects of all kinds of government policies can be analyzed, but it also makes the state space very big. A cross-sectional distribution as state variable requires new concepts (developed in measure theory) for defining and new computational techniques for computing equilibria. The early papers therefore restricted attention to steady state equilibria (in which the cross-sectional wealth distribution remained constant). Very recently techniques have been developed to handle economies with distributions as state variables that feature aggregate shocks, so that the cross-sectional wealth distribution itself varies over time. Krusell and Smith (1998) is the key reference. Applications of their techniques to interesting policy questions could be very rewarding in the future. If time permits I will discuss such an application due to Heathcote (1999).

So far we have not considered how government policies affect equilibrium allocations and prices. In the next modules this question is taken up. First we discuss fiscal policy and we start with positive questions: how does the governments' decision to finance a given stream of expenditures (debt vs. taxes) affect macroeconomic aggregates (Barro (1974), Ohanian (1997))?; how does government spending affect output (Baxter and King (1993))? In this discussion government policy is taken as exogenously given. The next question is of normative nature: how should a benevolent government carry out fiscal policy? The answer to this question depends crucially on the assumption of whether the government can commit to its policy. A government that can solves a classical Ramsey problem (not to be confused with the Ramsey model); the main results on optimal fiscal policy are reviewed in Chari and Kehoe (1999). Kydland and Prescott (1977) pointed out the dilemma a government faces if it cannot commit to its policy -this is the classical time consistency problem. How a benevolent government that cannot commit should carry out fiscal policy is still very much an open question. Klein and Rios-Rull (1999) have made substantial progress in answering this question. Note that we throughout our discussion assume that the government acts in the best interest of its citizens. What happens if policies are instead chosen by votes of selfish individuals is discussed in the last part of the course.

As discussed before we assumed so far that government policies were either fixed exogenously or set by a benevolent government (that can or can't commit). Now we relax this assumption and discuss political-economic equilibria in which people not only act rationally with respect to their economic decisions, but also rationally with respect to their voting decisions that determine macroeconomic policy. Obviously we first had to discuss models with heterogeneous agents since with homogeneous agents there is no political conflict and hence no interesting differences between the Ramsey problem and a political-economic equilibrium. This area of research is not very far developed and we will only present two examples (Krusell et al. (1997), Alesina and Rodrik (1994) that deal with the question of capital taxation in a dynamic general equilibrium model in which the capital tax rate is decided upon by repeated voting.

2 A Simple Dynamic Economy

2.1 General Principles for Specifying a Model

An economic model consists of different types of entities that take decisions subject to constraints. When writing down a model it is therefore crucial to clearly state what the agents of the model are, which decisions they take, what constraints they have and what information they possess when making their decisions. Typically a model has (at most) three types of decision-makers

1. Households: We have to specify households **preferences** over **commodities** and their **endowments** of these commodities. Households are assumed to maximize their preferences, subject to a constraint set that specifies which combination of commodities a household can choose from. This set usually depends on the initial endowments and on market prices.

2. Firms: We have to specify the **technology** available to firms, describing how commodities (inputs) can be transformed into other commodities (outputs). Firms are assumed to maximize (expected) profits, subject to their production plans being technologically feasible.

3. Government: We have to specify what **policy** instruments (taxes, money supply etc.) the government controls. When discussing government policy from a positive point of view we will take government polices as given (of course requiring the government budget constraint(s) to be satisfied), when discussing government policy from a normative point of view we will endow the government, as households and firms, with an objective function. The government will then maximize this objective function by choosing policy, subject to the policies satisfying the government budget constraint(s)).

In addition to specifying preferences, endowments, technology and policy, we have to specify what **information** agents possess when making decisions. This will become clearer once we discuss models with uncertainty. Finally we have to be precise about how agents interact with each other. Most of economics focuses on market interaction between agents; this will be also the case in this course. Therefore we have to specify our **equilibrium concept,** by making assumptions about how agents perceive their power to affect market prices. In this course we will focus on competitive equilibria, by assuming that all agents in the model (apart from possibly the government) take market prices as given and beyond their control when making their decisions. An alternative assumption would be to allow for market power of firms or households, which induces strategic interactions between agents in the model. Equilibria involving strategic interaction have to be analyzed using methods from modern game theory, which you will be taught in the second quarter of the micro sequence by Prof. Bernheim.

To summarize, a description of any model in this course should always contain the specification of the elements in bold letters: what commodities are traded, preferences over and endowments of these commodities, technology, government policies, the information structure and the equilibrium concept.

2.2 An Example Economy

Time is discrete and indexed by $t = 0, 1, 2, \ldots$ There are 2 individuals that live forever in this pure exchange economy. There are no firms or any government in this economy. In each period the two agents trade a nonstorable consumption good. Hence there are (countably) infinite number of commodities, namely consumption in periods $t = 0, 1, 2, \ldots$

Definition 1 *An allocation is a sequence* $(c^1, c^2) = \{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ *of consumption in each period for each individual.*

Individuals have preferences over consumption allocations that can be represented by the utility function

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{1}$$

with $\beta \in (0, 1)$.

This utility function satisfies some assumptions that we will often require in this course. These are further discussed in the appendix to this chapter. Note that both agents are assumed to have the same time discount factor β.

Agents have deterministic endowment streams $e^i = \{e_t^i\}_{t=0}^{\infty}$ of the consumption goods given by

$$e_t^1 = \begin{cases} 2 \text{ if } t \text{ is even} \\ 0 \text{ if } t \text{ is odd} \end{cases}$$

$$e_t^2 = \begin{cases} 0 \text{ if } t \text{ is even} \\ 2 \text{ if } t \text{ is odd} \end{cases}$$

There is no uncertainty in this model and both agents know their endowment pattern perfectly in advance. All information is public, i.e. all agents know everything. At period 0, before endowments are received and consumption takes place, the two agents meet at a central market place and trade all commodities, i.e. trade consumption for all future dates. Let p_t denote the price, in period 0, of one unit of consumption to be delivered in period t, in terms of an abstract unit of account. We will see later that prices are only determined up to a constant, so we can always normalize the price of one commodity to 1 and make it the numeraire. Both agents are assumed to behave competitively in that they take the sequence of prices $\{p_t\}_{t=0}^{\infty}$ as given and beyond their control when making their consumption decisions.

After trade has occurred agents possess pieces of paper (one may call them contracts) stating in period 212 I, agent 1, will eat 1.75 units of the consumption good and will deliver 0.25 units to agent 2

in period 2525 I, agent 1, will receive one unit of the consumption good from agent 2 (and eat it).

and so forth. In all future periods the only thing that happens is that agents meet at the market place again and deliveries of the consumption goods they agreed upon in period 0 takes place. Again, all trade takes place in period 0 and agents are committed in future periods to what they have agreed upon in period 0. In particular, there is perfect enforcement of the contracts signed in period 0.[1]

2.3 Definition of Competitive Equilibrium

Given a sequence of prices $\{p_t\}_{t=0}^{\infty}$ households then solve the following optimization problem

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

$$\text{s.t.}$$

$$\begin{aligned} \sum_{t=0}^{\infty} p_t c_t^i &\leq \sum_{t=0}^{\infty} p_t e_t^i \\ c_t^i &\geq 0 \text{ for all } t \end{aligned}$$

Note that the budget constraint can be rewritten as

$$\sum_{t=0}^{\infty} p_t (e_t^i - c_t^i) \geq 0$$

The quantity $e_t^i - c_t^i$ is the net trade of consumption of agent i for period t which may be positive or negative.

For arbitrary prices $\{p_t\}_{t=0}^{\infty}$ it may be the case that total consumption in the economy desired by both agents, $c_t^1 + c_t^2$ at these prices does not equal total endowments $e_t^1 + e_t^2 \equiv 2$. We will call equilibrium a situation in which prices are "right" in the sense that they induce agents to choose consumption so that total consumption equals total endowment in each period. More precisely, we have the following definition

Definition 2 *A (competitive) Arrow-Debreu equilibrium are prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ *such that*

[1] A market structure in which agents trade only at period 0 will be called an Arrow-Debreu market structure. We will show below that this market structure is equivalent to a market structure in which trade in consumption and a particular asset takes place in each period, a market structure that we will call sequential markets.

1. *Given* $\{\hat{p}_t\}_{t=0}^{\infty}$, *for* $i = 1, 2$, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ *solves*

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{2}$$

$$s.t.$$

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i \tag{3}$$

$$c_t^i \geq 0 \text{ for all } t \tag{4}$$

2.

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 \text{ for all } t \tag{5}$$

The elements of an equilibrium are allocations and prices. Note that we do not allow free disposal of goods, as the market clearing condition is stated as an equality.[2] Also note the ˆ's in the appropriate places: the consumption allocation has to satisfy the budget constraint (3) only at equilibrium prices and it is the equilibrium consumption allocation that satisfies the goods market clearing condition (5). Since in this course we will only talk about competitive equilibria, we will henceforth take the adjective "competitive" as being understood.

2.4 Solving for the Equilibrium

For arbitrary prices $\{p_t\}_{t=0}^{\infty}$ let's first solve the consumer problem. Attach the Lagrange multiplier λ_i to the budget constraint. The first order necessary conditions for c_t^i and c_{t+1}^i are then

$$\frac{\beta^t}{c_t^i} = \lambda_i p_t \tag{6}$$

$$\frac{\beta^{t+1}}{c_{t+1}^i} = \lambda_i p_{t+1} \tag{7}$$

and hence

$$p_{t+1} c_{t+1}^i = \beta p_t c_t^i \text{ for all } t \tag{8}$$

for $i = 1, 2$.

Equations (8), together with the budget constraint can be solved for the optimal sequence of consumption of household i as a function of the infinite sequence of prices (and of the endowments, of course)

$$c_t^i = c_t^i\left(\{p_t\}_{t=0}^{\infty}\right)$$

[2] Different people have different tastes as to whether one should allow free disposal or not. Personally I think that if one wishes to allow free disposal, one should specify this as part of technology (i.e. introduce a firm that has available a technology that uses positive inputs to produce zero output; obviously for such a firm to be operative in equilibrium it has to be the case that the price of the inputs are non-positive -think about goods that are actually bads such as pollution).

In order to solve for the equilibrium prices $\{p_t\}_{t=0}^{\infty}$ one then uses the goods market clearing conditions (5)

$$c_t^1\left(\{p_t\}_{t=0}^{\infty}\right) + c_t^2\left(\{p_t\}_{t=0}^{\infty}\right) = e_t^1 + e_t^2 \text{ for all } t$$

This is a system of infinite equations (for each t one) in an infinite number of unknowns $\{p_t\}_{t=0}^{\infty}$ which is in general hard to solve. Below we will discuss Negishi's method that often proves helpful to simplify solving for equilibria.

For our particular example economy things are more straightforward. Sum (8) across agents to obtain

$$p_{t+1}\left(c_{t+1}^1 + c_{t+1}^2\right) = \beta p_t(c_t^1 + c_t^2)$$

Using the goods market clearing condition we find that

$$p_{t+1}\left(e_{t+1}^1 + e_{t+1}^2\right) = \beta p_t(e_t^1 + e_t^2)$$

and hence

$$p_{t+1} = \beta p_t$$

and therefore equilibrium prices have to satisfy

$$p_t = \beta^t p_0$$

Without loss of generality we can set $p_0 = 1$, i.e. make consumption at period 0 the numeraire.[3] Then equilibrium prices have to satisfy

$$\hat{p}_t = \beta^t$$

so that, since $\beta < 1$, the period 0 price for period t consumption is lower than the period 0 price for period 0 consumption. This fact just reflects the impatience of both agents.

Using (8) we have that $c_{t+1}^i = c_t^i = c_0^i$ for all t, i.e. consumption is constant across time for both agents. This reflects the agent's desire to smooth consumption over time, a consequence of the strict concavity of the period utility function. Now observe that the budget constraint of both agents will hold with equality since agents' period utility function is strictly increasing. The left hand side becomes

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i = c_0^i \sum_{t=0}^{\infty} \beta^t = \frac{c_0^i}{1-\beta}$$

for $i = 1, 2$.

[3] Note that multiplying all prices by $\mu > 0$ does not change the budget constraints of agents, so that if prices $\{p_t\}_{t=0}^{\infty}$ and allocations $(\{c_t^i\}_{t=0}^{\infty})_{i\in 1,2}$ are an AD equilibrium, so is prices $\{\mu p_t\}_{t=0}^{\infty}$ and allocations $(\{c_t^i\}_{t=0}^{\infty})_{i=1,2}$

The two agents differ only along one dimension: agent 1 is rich first, which, given that prices are declining over time, is an advantage. For agent 1 the right hand side of the budget constraint becomes

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^1 = 2\sum_{t=0}^{\infty} \beta^{2t} = \frac{2}{1-\beta^2}$$

and for agent 2 it becomes

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^2 = 2\beta\sum_{t=0}^{\infty} \beta^{2t} = \frac{2\beta}{1-\beta^2}$$

The equilibrium allocation is then given by

$$\begin{aligned} \hat{c}_t^1 &= \hat{c}_0^1 = (1-\beta)\frac{2}{1-\beta^2} = \frac{2}{1+\beta} > 1 \\ \hat{c}_t^2 &= \hat{c}_0^2 = (1-\beta)\frac{2\beta}{1-\beta^2} = \frac{2\beta}{1+\beta} < 1 \end{aligned}$$

which obviously satisfies

$$\hat{c}_t^1 + \hat{c}_t^2 = 2 = \hat{e}_t^1 + \hat{e}_t^2 \text{ for all } t$$

Therefore the mere fact that the first agent is rich first makes her consume more in *every* period. Note that there is substantial trade going on; in each even period the first agent delivers $2 - \frac{2}{1+\beta} = \frac{2\beta}{1+\beta}$ to the second agent and in all odd periods the second agent delivers $2 - \frac{2\beta}{1+\beta}$ to the first agent. Also note that this trade is mutually beneficial, because without trade both agents receive lifetime utility

$$u(e_t^i) = -\infty$$

whereas with trade they obtain

$$\begin{aligned} u(\hat{c}^1) &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2}{1+\beta}\right) = \frac{\ln\left(\frac{2}{1+\beta}\right)}{1-\beta} > 0 \\ u(\hat{c}^2) &= \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2\beta}{1+\beta}\right) = \frac{\ln\left(\frac{2\beta}{1+\beta}\right)}{1-\beta} < 0 \end{aligned}$$

In the next section we will show that not only are both agents better off in the competitive equilibrium than by just eating their endowment, but that, in a sense to be made precise, the equilibrium consumption allocation is socially optimal.

2.5 Pareto Optimality and the First Welfare Theorem

In this section we will demonstrate that for this economy a competitive equilibrium is socially optimal. To do this we first have to define what socially optimal means. Out notion of optimality will be Pareto efficiency (also sometimes referred to as Pareto optimal). Loosely speaking, an allocation is Pareto efficient if it is feasible and if there is no other feasible allocation that makes no household worse off and at least one household strictly better off. Let us now make this precise.

Definition 3 *An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is feasible if*

1.
$$c_t^i \geq 0 \text{ for all } t, \text{ for } i = 1, 2$$

2.
$$c_t^1 + c_t^2 = e_t^1 + e_t^2 \text{ for all } t$$

Feasibility requires that consumption is nonnegative and satisfies the resource constraint for all periods $t = 0, 1, \ldots$

Definition 4 *An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is Pareto efficient if it is feasible and if there is no other feasible allocation $\{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}$ such that*

$$\begin{aligned} u(\tilde{c}^i) &\geq u(c^i) \text{ for both } i = 1, 2 \\ u(\tilde{c}^i) &> u(c^i) \text{ for at least one } i = 1, 2 \end{aligned}$$

Note that Pareto efficiency has nothing to do with fairness in any sense: an allocation in which agent 1 consumes everything in every period and agent 2 starves is Pareto efficient, since we can only make agent 2 better off by making agent 1 worse off.

We now prove that every competitive equilibrium allocation for the economy described above is Pareto efficient. Note that we have solved for one equilibrium above; this does not rule out that there is more than one equilibrium. One can, in fact, show that for this economy the competitive equilibrium is unique, but we will not pursue this here.

Proposition 5 *Let $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ be a competitive equilibrium allocation. Then $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ is Pareto efficient.*

Proof. The proof will be by contradiction; we will assume that $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ is not Pareto efficient and derive a contradiction to this assumption.

So suppose that $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ is not Pareto efficient. Then by the definition of Pareto efficiency there exists another feasible allocation $(\{\tilde{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ such that

$$\begin{aligned} u(\tilde{c}^i) &\geq u(\hat{c}^i) \text{ for both } i = 1, 2 \\ u(\tilde{c}^i) &> u(\hat{c}^i) \text{ for at least one } i = 1, 2 \end{aligned}$$

Without loss of generality assume that the strict inequality holds for $i = 1$.

Step 1: Show that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$

where $\{\hat{p}_t\}_{t=0}^{\infty}$ are the equilibrium prices associated with $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$. If not, i.e. if

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$

then for agent 1 the $\tilde{}$-allocation is better (remember $u(\tilde{c}^i) > u(\hat{c}^i)$ is assumed) and not more expensive, which cannot be the case since $\{\hat{c}_t^1\}_{t=0}^{\infty}$ is part of a competitive equilibrium, i.e. maximizes agent 1's utility given equilibrium prices. Hence

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 \tag{9}$$

Step 2: Show that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

If not, then

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

But then there exists a $\delta > 0$ such that

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

Remember that we normalized $\hat{p}_0 = 1$. Now define a new allocation for agent 2, by

$$\begin{aligned} \check{c}_t^2 &= \tilde{c}_t^2 \text{ for all } t \geq 1 \\ \check{c}_0^2 &= \tilde{c}_0^2 + \delta \text{ for } t = 0 \end{aligned}$$

Obviously

$$\sum_{t=0}^{\infty} \hat{p}_t \check{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

and

$$u(\check{c}) > u(\tilde{c}) \geq u(\hat{c})$$

which can't be the case since $\{\hat{c}_t^2\}_{t=0}^{\infty}$ is part of a competitive equilibrium, i.e. maximizes agent 2's utility given equilibrium prices. Hence

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 \tag{10}$$

Step 3: Now sum equations (9) and (10) to obtain

$$\sum_{t=0}^{\infty} \hat{p}_t(\tilde{c}_t^1 + \tilde{c}_t^2) > \sum_{t=0}^{\infty} \hat{p}_t(\hat{c}_t^1 + \hat{c}_t^2)$$

But since both allocations are feasible (the allocation $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ because it is an equilibrium allocation, the allocation $(\{\tilde{c}_t^i\}_{t=0}^{\infty})_{i=1,2}$ by assumption) we have that

$$\tilde{c}_t^1 + \tilde{c}_t^2 = e_t^1 + e_t^2 = \hat{c}_t^1 + \hat{c}_t^2 \text{ for all } t$$

and thus

$$\sum_{t=0}^{\infty} \hat{p}_t(e_t^1 + e_t^2) > \sum_{t=0}^{\infty} \hat{p}_t(e_t^1 + e_t^2),$$

our desired contradiction. ■

2.6 Negishi's (1960) Method to Compute Equilibria

In the example economy considered in this section it was straightforward to compute the competitive equilibrium by hand. This is usually not the case for dynamic general equilibrium models. Now we describe a method to compute equilibria for economies in which the welfare theorem(s) hold. The main idea is to compute Pareto-optimal allocations by solving an appropriate social planners problem. This social planner problem is a simple optimization problem which does not involve any prices (still infinite-dimensional, though) and hence much easier to tackle in general than a full-blown equilibrium analysis which consists of several optimization problems (one for each consumer) plus market clearing and involves allocations *and* prices. If the first welfare theorem holds then we know that competitive equilibrium allocations are Pareto optimal; by solving for all Pareto optimal allocations we have then solved for all potential equilibrium allocations. Negishi's method provides an algorithm to compute all Pareto optimal allocations and to isolate those who are in fact competitive equilibrium allocations.

We will repeatedly apply this trick in this course: solve a simple social planners problem and use the welfare theorems to argue that we have solved for the allocations of competitive equilibria. Then find equilibrium prices that support these allocations. The news is even better: usually we can read off the prices as Lagrange multipliers from the appropriate constraints of the social planners problem. In later parts of the course we will discuss economies in which the welfare theorems do not hold. We will see that these economies are much harder to analyze exactly because there is no simple optimization problem that completely characterizes the (set of) equilibria of these economies.

Consider the following social planners problem

$$\begin{aligned} & \max_{\{(c_t^1, c_t^2)\}_{t=0}^{\infty}} \alpha u(c^1) + (1-\alpha)u(c^2) \qquad (11) \\ = & \max_{\{(c_t^1, c_t^2)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\alpha \ln(c_t^1) + (1-\alpha)\ln(c_t^2)\right] \\ & \text{s.t.} \\ c_t^i \geq & \ 0 \text{ for all } i, \text{ all } t \\ c_t^1 + c_t^2 = & \ e_t^1 + e_t^2 \equiv 2 \text{ for all } t \end{aligned}$$

for a Pareto weight $\alpha \in [0,1]$. The social planner maximizes the weighted sum of utilities of the two agents, subject to the allocation being feasible. The weight α indicates how important agent 1's utility is to the planner, relative to agent 2's utility. Note that the solution to this problem depends on the Pareto weights, i.e. the optimal consumption choices are functions of α

$$\{(c_t^1, c_t^2)\}_{t=0}^{\infty} = \{(c_t^1(\alpha), c_t^2(\alpha))\}_{t=0}^{\infty}$$

We have the following

Proposition 6 *An allocation* $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ *is Pareto efficient if and only if it solves the social planners problem* (11) *for some* $\alpha \in [0,1]$

Proof. Omitted (but a good exercise) ■

This proposition shows that we can characterize the set of all Pareto efficient allocations by varying α between 0 and 1 and solving the social planners problem for all α's. As we will demonstrate, by choosing a particular α, the particular efficient allocation for that α turns out to be the competitive equilibrium allocation.

Now let us solve the planners problem for arbitrary $\alpha \in (0,1)$.[4] Attach Lagrange multipliers $\frac{\mu_t}{2}$ to the resource constraints (and ignore the non-negativity constraints on c_t^i since they never bind, due to the period utility function satisfying the Inada conditions). The reason why we divide by 2 will become apparent in a moment.

[4] Note that for $\alpha = 0$ and $\alpha = 1$ the solution to the problem is trivial. For $\alpha = 0$ we have $c_t^1 = 0$ and $c_t^2 = 2$ and for $\alpha = 1$ we have the reverse.

The first order necessary conditions are

$$\begin{aligned}\frac{\alpha\beta^t}{c_t^1} &= \frac{\mu_t}{2} \\ \frac{(1-\alpha)\beta^t}{c_t^2} &= \frac{\mu_t}{2}\end{aligned}$$

Combining yields

$$\frac{c_t^1}{c_t^2} = \frac{\alpha}{1-\alpha} \tag{12}$$

$$c_t^1 = \frac{\alpha}{1-\alpha} c_t^2 \tag{13}$$

i.e. the ratio of consumption between the two agents equals the ratio of the Pareto weights in every period t. A higher Pareto weight for agent 1 results in this agent receiving more consumption in every period, relative to agent 2. Using the resource constraint in conjunction with (13) yields

$$\begin{aligned}c_t^1 + c_t^2 &= 2 \\ \frac{\alpha}{1-\alpha} c_t^2 + c_t^2 &= 2 \\ c_t^2 &= 2(1-\alpha) = c_t^2(\alpha) \\ c_t^1 &= 2\alpha = c_t^1(\alpha)\end{aligned}$$

i.e. the social planner divides the total resources in every period according to the Pareto weights. Note that the division is the same in every period, independent of the agents endowments in that particular period. The Lagrange multipliers are given by

$$\mu_t = \frac{2\alpha\beta^t}{c_t^1} = \beta^t$$

(if we wouldn't have done the initial division by 2 we would have to carry the $\frac{1}{2}$ around from now on; the results below wouldn't change at all).

Hence for this economy the set of Pareto efficient allocations is given by

$$PO = \{\{(c_t^1, c_t^2)\}_{t=0}^{\infty} : c_t^1 = 2\alpha \text{ and } c_t^2 = 2(1-\alpha) \text{ for some } \alpha \in [0,1]\}$$

How does this help us in finding the competitive equilibrium for this economy? Compare the first order condition of the social planners problem for agent 1

$$\frac{\alpha\beta^t}{c_t^1} = \frac{\mu_t}{2}$$

or

$$\frac{\beta^t}{c_t^1} = \frac{\mu_t}{2\alpha}$$

with the first order condition from the competitive equilibrium above (see equation (6)):

$$\frac{\beta^t}{c_t^1} = \lambda_1 p_t$$

By picking $\lambda_1 = \frac{1}{2\alpha}$ and $p_t = \beta^t$ these first order conditions are identical. Similarly, pick $\lambda_2 = \frac{1}{2(1-\alpha)}$ and one sees that the same is true for agent 2. So for appropriate choices of the individual Lagrange multipliers λ_i and prices p_t the optimality conditions for the social planners' problem and for the household maximization problems coincide. Resource feasibility is required in the competitive equilibrium as well as in the planners problem. Given that we found a unique equilibrium above but a lot of Pareto efficient allocations (for each α one), there must be an additional requirement that a competitive equilibrium imposes which the planners problem does not require.

In a competitive equilibrium households' choices are constrained by the *budget constraint;* the planner is only concerned with resource balance. The last step to single out competitive equilibrium allocations from the set of Pareto efficient allocations is to ask which Pareto efficient allocations would be affordable for all households if these holds were to face as market prices the Lagrange multipliers from the planners problem (that the Lagrange multipliers are the appropriate prices is harder to establish, so let's proceed on faith for now). Define the transfer functions $t^i(\alpha)$, $i = 1, 2$ by

$$t^i(\alpha) = \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right]$$

The number $t^i(\alpha)$ is the amount of the numeraire good (we pick the period 0 consumption good) that agent i would need as transfer in order to be able to afford the Pareto efficient allocation indexed by α. One can show that the t^i as functions of α are homogeneous of degree one[5] and sum to 0 (see HW 1).

Computing $t^i(\alpha)$ for the current economy yields

$$\begin{aligned} t^1(\alpha) &= \sum_t \mu_t \left[c_t^1(\alpha) - e_t^1\right] \\ &= \sum_t \beta^t \left[2\alpha - e_t^1\right] \\ &= \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2} \\ t^2(\alpha) &= \frac{2(1-\alpha)}{1-\beta} - \frac{2\beta}{1-\beta^2} \end{aligned}$$

To find the competitive equilibrium allocation we now need to find the Pareto weight α such that $t^1(\alpha) = t^2(\alpha) = 0$, i.e. the Pareto optimal allocation that

[5] In the sense that if one gives weight $x\alpha$ to agent 1 and $x(1-\alpha)$ to agent 2, then the corresponding required transfers are xt^1 and xt^2.

both agents can afford with zero transfers. This yields

$$\begin{aligned} 0 &= \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2} \\ \alpha &= \frac{1}{1+\beta} > 1 \end{aligned}$$

and the corresponding allocations are

$$\begin{aligned} c_t^1\left(\frac{1}{1+\beta}\right) &= \frac{2}{1+\beta} \\ c_t^2\left(\frac{1}{1+\beta}\right) &= \frac{2\beta}{1+\beta} \end{aligned}$$

Hence we have solved for the equilibrium allocations; equilibrium prices are given by the Lagrange multipliers $\mu_t = \beta^t$ (note that without the normalization by $\frac{1}{2}$ at the beginning we would have found the same allocations and equilibrium prices $p_t = \frac{\beta^t}{2}$ which, given that equilibrium prices are homogeneous of degree 0, is perfectly fine, too).

To summarize, to compute competitive equilibria using Negishi's method one does the following

1. Solve the social planners problem for Pareto efficient allocations indexed by Pareto weight α

2. Compute transfers, indexed by α, necessary to make the efficient allocation affordable. As prices use Lagrange multipliers on the resource constraints in the planners' problem.

3. Find the Pareto weight(s) $\hat{\alpha}$ that makes the transfer functions 0.

4. The Pareto efficient allocations corresponding to $\hat{\alpha}$ are equilibrium allocations; the supporting equilibrium prices are (multiples of) the Lagrange multipliers from the planning problem

Remember from above that to solve for the equilibrium directly in general involves solving an infinite number of equations in an infinite number of unknowns. The Negishi method reduces the computation of equilibrium to a finite number of equations in a finite number of unknowns in step 3 above. For an economy with two agents, it is just one equation in one unknown, for an economy with N agents it is a system of $N-1$ equations in $N-1$ unknowns. This is why the Negishi method (and methods relying on solving appropriate social planners problems in general) often significantly simplifies solving for competitive equilibria.

2.7 Sequential Markets Equilibrium

The market structure of Arrow-Debreu equilibrium in which all agents meet only once, at the beginning of time, to trade claims to future consumption may seem empirically implausible. In this section we show that the same allocations as in an Arrow-Debreu equilibrium would arise if we let agents trade consumption and one-period bonds in each period. We will call a market structure in which markets for consumption and assets open in each period Sequential Markets and the corresponding equilibrium Sequential Markets (SM) equilibrium.[6]

Let r_{t+1} denote the interest rate on one period bonds from period t to period $t+1$. A one period bond is a promise (contract) to pay 1 unit of the consumption good in period $t+1$ in exchange for $\frac{1}{1+r_{t+1}}$ units of the consumption good in period t. We can interpret $q_t \equiv \frac{1}{1+r_{t+1}}$ as the relative price of one unit of the consumption good in period $t+1$ in terms of the period t consumption good. Let a_{t+1}^i denote the amount of such bonds purchased by agent i in period t and carried over to period $t+1$. If $a_{t+1}^i < 0$ we can interpret this as the agent taking out a one-period loan at interest rate r_{t+1}. Household i's budget constraint in period t reads as

$$c_t^i + \frac{a_{t+1}^i}{(1+r_{t+1})} \leq e_t^i + a_t^i \tag{14}$$

or

$$c_t^i + q_t a_{t+1}^i \leq e_t^i + a_t^i$$

Agents start out their life with initial bond holdings a_0^i (remember that period 0 bonds are claims to period 0 consumption). Mostly we will focus on the situation in which $a_0^i = 0$ for all i, but sometimes we want to start an agent off with initial wealth ($a_0^i > 0$) or initial debt ($a_0^i < 0$). We then have the following definition

Definition 7 *A Sequential Markets equilibrium is allocations* $\left\{\left(\hat{c}_t^i, \hat{a}_{t+1}^i\right)_{i=1,2}\right\}_{t=1}^{\infty}$, *interest rates* $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ *such that*

1. *For* $i = 1, 2$, *given interest rates* $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ $\{\hat{c}_t^i, \hat{a}_{t+1}^i\}_{t=0}^{\infty}$ *solves*

$$\max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{15}$$

s.t.

$$c_t^i + \frac{a_{t+1}^i}{(1+r_{t+1})} \leq e_t^i + a_t^i \tag{16}$$

$$c_t^i \geq 0 \text{ for all } t \tag{17}$$

$$a_{t+1}^i \geq -\bar{A}^i \tag{18}$$

[6] In the simple model we consider in this section the restriction of assets traded to one-period riskless bonds is without loss of generality. In more complicated economies (with uncertainty, say) it would not be. We will come back to this issue in later chapters.

2. *For all* $t \geq 0$

$$\begin{aligned} \sum_{i=1}^{2} \hat{c}_t^i &= \sum_{i=1}^{2} e_t^i \\ \sum_{i=1}^{2} \hat{a}_{t+1}^i &= 0 \end{aligned}$$

The constraint (18) on borrowing is necessary to guarantee existence of equilibrium. Suppose that agents would not face any constraint as to how much they can borrow, i.e. suppose the constraint (18) were absent. Suppose there would exist a AM-equilibrium $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i=1,2}\}_{t=1}^{\infty}, \{\hat{r}_{t+1}\}_{t=0}^{\infty}$. Without constraint on borrowing agent i could always do better by setting

$$\begin{aligned} c_0^i &= \hat{c}_0^i + \frac{\varepsilon}{1+\hat{r}_1} \\ a_1^i &= \hat{a}_1^i - \varepsilon \\ a_2^i &= \hat{a}_2^i - (1+\hat{r}_2)\varepsilon \\ a_{t+1}^i &= \hat{a}_{t+1}^i - \prod_{t=1}^{t}(1+\hat{r}_{t+1})\varepsilon \end{aligned}$$

i.e. by borrowing $\varepsilon > 0$ more in period 0, consuming it and then rolling over the additional debt forever, by borrowing more and more. Such a scheme is often called a Ponzi scheme. Hence without a limit on borrowing no SM equilibrium can exist because agents would run Ponzi schemes.

In this section we are interested in specifying a borrowing limit that prevents Ponzi schemes, yet is high enough so that households are never constrained in the amount they can borrow (by this we mean that a household, knowing that it can not run a Ponzi scheme, would always find it optimal to choose $a_{t+1}^i > -\bar{A}^i$). In later chapters we will analyze economies in which agents face borrowing constraints that are binding in certain situations. Not only are SM equilibria for these economies quite different from the ones to be studied here, but also the equivalence between SM equilibria and AD equilibria will break down.

We are now ready to state the equivalence theorem relating AD equilibria and SM equilibria. Assume that $a_0^i = 0$ for all $i = 1, 2$.

Proposition 8 *Let allocations* $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ *and prices* $\{\hat{p}_t\}_{t=0}^{\infty}$ *form an Arrow-Debreu equilibrium. Then there exist* $(\bar{A}^i)_{i=1,2}$ *and a corresponding sequential markets equilibrium with allocations* $\{(\tilde{c}_t^i, \tilde{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ *such that*

$$\tilde{c}_t^i = \hat{c}_t^i \text{ for all } i, \text{ all } t$$

Reversely, let allocations $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ *and interest rates* $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ *form a sequential markets equilibrium. Suppose that it satisfies*

$$\begin{aligned} \hat{a}_{t+1}^i &> -\bar{A}^i \text{ for all } i, \text{ all } t \\ \hat{r}_{t+1} &> 0 \text{ for all } t \end{aligned}$$

Then there exists a corresponding Arrow-Debreu equilibrium $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}, \{\tilde{p}_t\}_{t=0}^{\infty}$ *such that*

$$\hat{c}_t^i = \tilde{c}_t^i \quad \text{for all } i, \text{ all } t$$

Proof. Step 1: The key to the proof is to show the equivalence of the budget sets for the Arrow-Debreu and the sequential markets structure. Normalize $\hat{p}_0 = 1$ and relate equilibrium prices and interest rates by

$$1 + \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}} \tag{19}$$

Now look at the sequence of sequential markets budget constraints and assume that they hold with equality (which they do in equilibrium, due to the nonsatiation assumption)

$$c_0^i + \frac{a_1^i}{1+\hat{r}_1} = e_0^i \tag{20}$$

$$c_1^i + \frac{a_2^i}{1+\hat{r}_2} = e_1^i + a_1^i \tag{21}$$

$$\vdots$$

$$c_t^i + \frac{a_{t+1}^i}{1+\hat{r}_{t+1}} = e_t^i + a_t^i \tag{22}$$

Substituting for a_1^i from (21) in (20) one gets

$$c_0^i + \frac{c_1^i}{1+\hat{r}_1} + \frac{a_2^i}{(1+\hat{r}_1)(1+\hat{r}_2)} = e_0^i + \frac{e_1^i}{(1+\hat{r}_1)}$$

and, repeating this exercise, one gets[7]

$$\sum_{t=0}^{T} \frac{c_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)} + \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{T} \frac{e_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)}$$

[7] We define

$$\prod_{j=1}^{0}(1+\hat{r}_j) = 1$$

Now note that (using the normalization $\hat{p}_0 = 1$)

$$\prod_{j=1}^{t}(1+\hat{r}_j) = \frac{\hat{p}_0}{\hat{p}_1} * \frac{\hat{p}_1}{\hat{p}_2} \cdots * \frac{\hat{p}_{t-1}}{\hat{p}_t} = \frac{1}{\hat{p}_t} \tag{23}$$

Taking limits with respect to t on both sides gives, using (23)

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i + \lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$

Given our assumptions on the equilibrium interest rates we have

$$\lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} \geq \lim_{T\to\infty} \frac{-\bar{A}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = 0$$

and hence

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$

Step 2: Now suppose we have an AD-equilibrium $\{(\hat{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$, $\{\hat{p}_t\}_{t=0}^{\infty}$. We want to show that there exist a SM equilibrium with same consumption allocation, i.e.

$$\tilde{c}_t^i = \hat{c}_t^i \text{ for all } i, \text{ all } t$$

Obviously $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ satisfies market clearing. Defining as asset holdings

$$\begin{aligned} \tilde{a}_{t+1}^i &= \sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau}\left(\hat{c}_{t+\tau}^i - e_{t+\tau}^i\right)}{\hat{p}_{t+1}} \\ &\geq -\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} e_{t+\tau}^i}{\hat{p}_{t+1}} \geq -\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} e_{t+\tau}^i}{\hat{p}_{t+1}} \geq -\sum_{\tau=0}^{\infty} \hat{p}_t e_t^i \end{aligned}$$

we see that the allocation satisfies the SM budget constraints (remember $1 + \tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$) Also note that

$$\tilde{a}_{t+1}^i > -\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} e_{t+\tau}^i}{\hat{p}_{t+1}} \geq -\sum_{\tau=1}^{\infty} \frac{\hat{p}_{t+\tau} e_{t+\tau}^i}{\hat{p}_{t+1}} \geq -\sum_{\tau=0}^{\infty} \hat{p}_t e_t^i > -\infty$$

so that we can take

$$\bar{A}^i = \sum_{\tau=0}^{\infty} \hat{p}_t e_t^i$$

This borrowing constraint, equalling the value of the endowment of agent i at AD-equilibrium prices is also called the natural debt limit. This borrowing limit is so high that agent i, knowing that she can't run a Ponzi scheme, will never reach it.

It remains to argue that $\{(\tilde{c}_t^i)_{i=1,2}\}_{t=0}^{\infty}$ maximizes utility, subject to the sequential markets budget constraints and the borrowing constraints. Take any other allocation satisfying these constraints. In step 1. we showed that this allocation satisfies the AD budget constraint. If it would be better than $\{\tilde{c}_t^i = \hat{c}_t^i\}_{t=0}^{\infty}$ it would have been chosen as part of an AD-equilibrium, which it wasn't. Hence $\{\tilde{c}_t^i\}_{t=0}^{\infty}$ is optimal within the set of allocations satisfying the SM budget constraints at interest rates $1 + \tilde{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$.

Step 3: Now suppose $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i \in I}\}_{t=1}^{\infty}$ and $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ form a sequential markets equilibrium satisfying

$$\begin{aligned} \hat{a}_{t+1}^i &> -\bar{A}^i \text{ for all } i, \text{ all } t \\ \hat{r}_{t+1} &> 0 \text{ for all } t \end{aligned}$$

We want to show that there exists a corresponding Arrow-Debreu equilibrium $\{(\tilde{c}_t^i)_{i \in I}\}_{t=0}^{\infty}$, $\{\tilde{p}_t\}_{t=0}^{\infty}$ with

$$\hat{c}_t^i = \tilde{c}_t^i \text{ for all } i, \text{ all } t$$

Again obviously $\{(\tilde{c}_t^i)_{i \in I}\}_{t=0}^{\infty}$ satisfies market clearing and, as shown in step 1, the AD budget constraint. It remains to be shown that it maximizes utility within the set of allocations satisfying the AD budget constraint. For $\tilde{p}_0 = 1$ and $\tilde{p}_{t+1} = \frac{\tilde{p}_t}{1+\hat{r}_{t+1}}$ the set of allocations satisfying the AD budget constraint coincides with the set of allocations satisfying the SM-budget constraint (for appropriate choices of asset holdings). Since in the SM equilibrium we have the additional borrowing constraints, the set over which we maximize in the AD case is larger, since the borrowing constraints are absent in the AD formulation. But by assumption these additional constraints are never binding ($\hat{a}_{t+1}^i > -\bar{A}^i$). Then from a basic theorem of constrained optimization we know that if the additional constraints are never binding, then the maximizer of the constrained problem is also the maximizer of the unconstrained problem, and hence $\{\tilde{c}_t^i\}_{t=0}^{\infty}$ is optimal for household i within the set of allocations satisfying her AD budget constraint. ■

This proposition shows that the sequential markets and the Arrow-Debreu market structures lead to identical equilibria, provided that we choose the no Ponzi conditions appropriately (equal to the natural debt limits, for example) and that the equilibrium interest rates are sufficiently high.[8] Usually the analysis of our economies is easier to carry out using AD language, but the SM formulation has more empirical appeal. The preceding theorem shows that we can have the best of both worlds.

[8]This assumption can be sufficiently weakened if one introduces borrowing constraints of slightly different form in the SM equilibrium to prevent Ponzi schemes. We may come back to this later.

For our example economy we find that the equilibrium interest rates in the SM formulation are given by

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}} = \frac{1}{\beta}$$

or

$$r_{t+1} = r = \frac{1}{\beta} - 1 = \rho$$

i.e. the interest rate is constant and equal to the subjective time discount rate.

2.8 Appendix

The utility function

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) \tag{24}$$

described in the main text satisfies the following assumptions that we will often require in our models:

1. Time separability: total utility from a consumption allocation c^i equals the discounted sum of period (or instantaneous) utility $U(c_t^i) = \ln(c_t^i)$. In particular, the period utility at time t only depends on consumption in period t and not on consumption in other periods. This formulation rules out, among other things, habit persistence.

2. Time discounting: the fact that $\beta < 1$ indicates that agents are impatient. The same amount of consumption yields less utility if it comes at a later time in an agents' life. The parameter β is often referred to as (subjective) time discount factor. The subjective time discount rate ρ is defined by $\beta = \frac{1}{1+\rho}$ and is often, as we will see, intimately related to the equilibrium interest rate in the economy (because the interest rate is nothing else but the market time discount rate).

3. Homotheticity: Define the marginal rate of substitution between consumption at any two dates t and $t+s$ as

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}}$$

The function u is said to be homothetic if $MRS(c_{t+s}, c_t) = MRS(\lambda c_{t+s}, \lambda c_t)$ for all $\lambda > 0$ and c. It is easy to verify that for u defined above we have

$$MRS(c_{t+s}, c_t) = \frac{\frac{\beta^{t+s}}{c_{t+s}}}{\frac{\beta^t}{c_t}} = \frac{\frac{\lambda\beta^{t+s}}{c_{t+s}}}{\frac{\lambda\beta^t}{c_t}} = MRS(\lambda c_{t+s}, \lambda c_t)$$

and hence u is homothetic. This, in particular, implies that if an agent's lifetime income doubles, optimal consumption choices will double in *each* period (income expansion paths are linear).[9] It also means that consumption allocations are independent of the units of measurement employed.

4. The instantaneous utility function or felicity function $U(c) = \ln(c)$ is continuous, twice continuously differentiable, strictly increasing (i.e. $U'(c) > 0$) and strictly concave (i.e. $U''(c) < 0$) and satisfies the Inada conditions

$$\begin{aligned} \lim_{c \searrow 0} U'(c) &= +\infty \\ \lim_{c \nearrow +\infty} U'(c) &= 0 \end{aligned}$$

These assumptions imply that more consumption is always better, but an additional unit of consumption yields less and less additional utility. The Inada conditions indicate that the first unit of consumption yields a lot of additional utility but that as consumption goes to infinity, an additional unit is (almost) worthless. The Inada conditions will guarantee that an agent always chooses $c_t \in (0, \infty)$ for all t

5. The felicity function U is a member of the class of Constant Relative Risk Aversion (CRRA) utility functions. These functions have the following important properties. First, define as $\sigma(c) = -\frac{U''(c)c}{U'(c)}$ the (Arrow-Pratt) coefficient of relative risk aversion. Hence $\sigma(c)$ indicates a household's attitude towards risk, with higher $\sigma(c)$ representing higher risk aversion. For CRRA utility functions $\sigma(c)$ is constant for all levels of consumption, and for $U(c) = \ln(c)$ it is not only constant, but equal to $\sigma(c) = \sigma = 1$. Second, the inverse of the intertemporal elasticity of substitution is_t measures by how many percent the slope of the indifference curve between c_t and c_{t+1}, $-\frac{U'(c_t)}{\beta U'(c_{t+1})}$ changes as the ratio between consumption tomorrow and today, $\frac{c_{t+1}}{c_t}$ changes by one percent, i.e.

$$(is_t)^{-1} = \frac{\left[\frac{d\left(-\frac{U'(c_t)}{\beta U'(c_{t+1})}\right)}{d\left(\frac{c_{t+1}}{c_t}\right)} \right]}{\left[\frac{-\frac{U'(c_t)}{\beta U'(c_{t+1})}}{\frac{c_{t+1}}{c_t}} \right]}$$

For $U(c) = \ln(c)$ utility this becomes, since $-\frac{U'(c_t)}{\beta U'(c_{t+1})} = -\frac{1}{\beta}\left(\frac{c_t}{c_{t+1}}\right)^{-1} = -\frac{1}{\beta}\frac{c_{t+1}}{c_t}$

$$(is_t)^{-1} = 1$$

[9] In the absense of borrowing constraints and other frictions which we will discuss later.

Therefore logarithmic period utility is sometimes also called isoelastic utility.[10] Hence for logarithmic period utility the intertemporal elasticity substitution is equal to the inverse of the coefficient of relative risk aversion.

[10]In general CRRA utility functions are of the form

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

and one can easily compute that the coefficient of relative risk aversion for this utility function is σ and the intertemporal elasticity of substitution equals σ^{-1}.

In a homework you will show that

$$\ln(c) = \lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

i.e. that logarithmic utility is a special case of this general class of utility functions.