7 Pareto Optima and Competitive Equilibria: The Two Welfare Theorems

In this section we will present the two fundamental theorems of welfare economics for economies in which the commodity space is a general (real) vector space, which is not necessarily finite dimensional. Since in macroeconomics we often deal with agents or economies that live forever, usually a finite dimensional commodity space is not sufficient for our analysis. The significance of the welfare theorems, apart from providing a normative justification for studying competitive equilibria is that planning problems characterizing Pareto optima are usually easier to solve that equilibrium problems, the ultimate goal of our theorizing.

Our discussion will follow Stokey et al. (1989), which in turn draws heavily on results developed by Debreu (1954).

7.1 What is an Economy?

We first discuss how what an economy is in Arrow-Debreu language. An economy $E = ((X_i, u_i)_{i \in I}, (Y_j)_{j \in J})$ consists of the following elements

1. A list of commodities, represented by the commodity space S. We require S to be a normed (real) vector space with norm $\|.\|$.[31]

[31] For completeness we state the following definitions

Definition 51 *A real vector space is a set S (whose elements are called vectors) on which are defined two operations*

- *Addition $+ : S \times S \to S$. For any $x, y \in S$, $x + y \in S$.*
- *Scalar Multiplication $\cdot : \mathbf{R} \times S \to S$. For any $\alpha \in \mathbf{R}$ and any $x \in S$, $\alpha x \in S$ that satisfy the following algebraic properties: for all $x, y \in S$ and all $\alpha, \beta \in \mathbf{R}$*
  - *(a)* $x + y = y + x$
  - *(b)* $(x + y) + z = x + (y + z)$
  - *(c)* $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$
  - *(d)* $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$
  - *(e)* $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$
  - *(f) There exists a null element $\theta \in S$ such that*

$$\begin{aligned} x + \theta &= x \\ 0 \cdot x &= \theta \end{aligned}$$

  - *(g)* $1 \cdot x = x$

Definition 52 *A normed vector space is a vector space is a vector space S together with a norm $\|.\| : S \to \mathbf{R}$ such that for all $x, y \in S$ and $\alpha \in \mathbf{R}$*

(a) $\|x\| \geq 0$, *with equality if and only if* $x = \theta$

(b) $\|\alpha \cdot x\| = |\alpha| \, \|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\|$

2. A finite set of people $i \in I$. Abusing notation I will by I denote both the set of people and the number of people in the economy.

3. Consumption sets $X_i \subseteq S$ for all $i \in I$. We will incorporate the restrictions that households endowments place on the x_i in the description of the consumption sets X_i.

4. Preferences representable by utility functions $u_i : S \to \mathbf{R}$.

5. A finite set of firms $j \in J$. The same remark about notation as above applies.

6. Technology sets $Y_j \subseteq S$ for all $j \in J$. Let by

$$Y = \sum_{j \in J} Y_j = \left\{ y \in S : \exists (y_j)_{j \in J} \text{ such that } y = \sum_{j \in J} y_j \text{ and } y_j \in Y_j \text{ for all } j \in J \right\}$$

denote the aggregate production set.

A private ownership economy $\tilde{E} = ((X_i, u_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J})$ consists of all the elements of an economy and a specification of ownership of the firms $\theta_{ij} \geq 0$ with $\sum_{i \in I} \theta_{ij} = 1$ for all $j \in J$. The entity θ_{ij} is interpreted as the share of ownership of household ι to firm j, i.e. the fraction of total profits of firm j that household i is entitled to.

With our formalization of the economy we can also make precise what we mean by an externality. An economy is said to exhibit an externality if household i's consumption set X_i or firm j's production set Y_j is affected by the choice of household k's consumption bundle x_k or firm m's production plan y_m. Unless otherwise stated we assume that we deal with an economy without externalities.

Definition 53 *An allocation is a tuple* $[(x_i)_{i \in I}, (y_j)_{j \in J}] \in S^{I \times J}$.

In the economy people supply factors of production and demand final output goods. We follow Debreu and use the convention that negative components of the x_i's denote factor inputs and positive components denote final goods. Similarly negative components of the y_j's denote factor inputs of firms and positive components denote final output of firms.

Definition 54 *An allocation* $[(x_i)_{i \in I}, (y_j)_{j \in J}] \in S^{I \times J}$ *is feasible if*

Note that in the first definition the adjective real refers to the fact that scalar multiplication is done with respect to a real number. Also note the intimate relation between a norm and a metric defined above. A norm of a vector space S, $\|.\| : S \to \mathbf{R}$ induces a metric $d : S \times S \to \mathbf{R}$ by

$$d(x, y) = \|x - y\|$$

1. $x_i \in X_i$ *for all* $i \in I$

2. $y_j \in Y_j$ *for all* $j \in J$

3. *(Resource Balance)*

$$\sum_{i \in I} x_i = \sum_{j \in J} y_j$$

Note that we require resource balance to hold with equality, ruling out free disposal. If we want to allow free disposal we will specify this directly as part of the description of technology.

Definition 55 *An allocation* $[(x_i)_{i \in I}, (y_j)_{j \in J}]$ *is Pareto optimal if*

1. *it is feasible*

2. *there does not exist another feasible allocation* $[(x_i^*)_{i \in I}, (y_j^*)_{j \in J}]$ *such that*

$$\begin{aligned} u_i(x_i^*) &\geq u_i(x_i) \text{ for all } i \in I \\ u_i(x_i^*) &> u_i(x_i) \text{ for at least one } i \in I \end{aligned}$$

Note that if $I = J = 1$ then[32] for an allocation $[x, y]$ resource balance requires $x = y$, the allocation is feasible if $x \in X \cap Y$, and the allocation is Pareto optimal if

$$x \in \arg \max_{z \in X \cap Y} u(z)$$

Also note that the definition of feasibility and Pareto optimality are identical for economies E and private ownership economies $\tilde{E}$. The difference comes in the definition of competitive equilibrium and there in particular in the formulation of the resource constraint. The discussion of competitive equilibrium requires a discussion of prices at which allocations are evaluated. Since we deal with possibly infinite dimensional commodity spaces, prices in general cannot be represented by a finite dimensional vector. To discuss prices for our general environment we need a more general notion of a price system. This is necessary in order to state and prove the welfare theorems for infinitely lived economies that we are interested in.

[32] The assumption that $J = 1$ is not at all restrictive if we restrict our attention to constant returns to scale technologies. Then, in any competitive equilibrium profits are zero and the number of firms is indeterminate in equilibrium; without loss of generality we then can restrict attention to a single representative firm. If we furthermore restrict attention to identical people and type identical allocations, then de facto $I = 1$. Under which assumptions the restriction to type identical allocations is justified will be discussed below.

7.2 Dual Spaces

A price system attaches to every bundle of the commodity space S a real number that indicates how much this bundle costs. If the commodity space is a finite (say $k-$) dimensional Euclidean space, then the natural thing to do is to represent a price system by a k-dimensional vector $p = (p_1, \ldots p_k)$, where p_l is the price of the l-th component of a commodity vector. The price of an entire point of the commodity space is then $\phi(s) = \sum_{l=1}^{k} s_l p_l$. Note that every $p \in \mathbf{R}^k$ represents a function that maps $S = \mathbf{R}^k$ into $\mathbf{R}$. Obviously, since for a given p and all $s, s' \in S$ and all $\alpha, \beta \in \mathbf{R}$

$$\phi(\alpha s + \beta s') = \sum_{l=1}^{k} p_l(\alpha s_l + \beta s'_l) = \alpha \sum_{l=1}^{k} p_l s_l + \beta \sum_{l=1}^{k} p_l s'_l = \alpha\phi(s) + \beta\phi(s')$$

the mapping associated with p is linear. We will take as a price system for an arbitrary commodity space S a continuous linear functional defined on S. The next definition makes the notion of a continuous linear functional precise.

Definition 56 *A linear functional ϕ on a normed vector space S (with associated norm $\|\|_S$) is a function $\phi : S \to \mathbf{R}$ that maps S into the reals and satisfies*

$$\phi(\alpha s + \beta s') = \alpha\phi(s) + \beta\phi(s') \text{ for all } s, s' \in S, \text{ all } \alpha, \beta \in \mathbf{R}$$

The functional ϕ is continuous if $\|s_n - s\|_S \to 0$ implies $|\phi(s_n) - \phi(s)| \to 0$ for all $\{s_n\}_{n=0}^{\infty} \in S, s \in S$. The functional ϕ is bounded if there exists a constant $M \in \mathbf{R}$ such that $|\phi(s)| \leq M \|s\|_S$ for all $s \in S$. For a bounded linear functional ϕ we define its norm by

$$\|\phi\|_d = \sup_{\|s\|_S \leq 1} |\phi(s)|$$

Fortunately it is rather easy to verify whether a linear functional is continuous and bounded. Stokey et al. state and prove a theorem that states that a linear functional is continuous if it is continuous at a particular point $s \in S$ and that it is bounded if (and only if) it is continuous. Hence a linear functional is bounded and continuous if it is continuous at a single point.

For any normed vector space S the space

$$S^* = \{\phi : \phi \text{ is a continuous linear functional on } S\}$$

is called the (algebraic) dual (or conjugate) space of S. With addition and scalar multiplication defined in the standard way S^* is a vector space, and with the norm $\|\|_d$ defined above S^* is a normed vector space as well. Note (you should prove this[33]) that even if S is not a complete space, S^* is a complete space and hence a Banach space (a complete normed vector space). Let us consider several examples that will be of interest for our economic applications.

[33] *After* you are done with this, check Kolmogorov and Fomin (1970), p. 187 (Theorem 1) for their proof.

Example 57 *For each* $p \in [1, \infty)$ *define the space* l_p *by*

$$l_p = \{x = \{x_t\}_{t=0}^{\infty} : x_t \in \mathbf{R}, \text{ for all } t; \ \|x\|_p = \left(\sum_{t=0}^{\infty} |x_t|^p \right)^{\frac{1}{p}} < \infty\}$$

with corresponding norm $\|x\|_p$. *For* $p = \infty$, *the space* l_∞ *is defined correspondingly, with norm* $\|x\|_\infty = \sup_t |x_t|$. *For any* $p \in [1, \infty)$ *define the conjugate index* q *by*

$$\frac{1}{p} + \frac{1}{q} = 1$$

For $p = 1$ *we define* $q = \infty$. *We have the important result that for any* $p \in [1, \infty)$ *the dual of* l_p *is* l_q. *This result can be proved by using the following theorem (which in turn is proved by Luenberger (1969), p. 107.)*

Theorem 58 *Every continuous linear functional* ϕ *on* l_p, $p \in [1, \infty)$, *is representable uniquely in the form*

$$\phi(x) = \sum_{t=0}^{\infty} x_t y_t \tag{38}$$

where $y = \{y_t\} \in l_q$. *Furthermore, every element of* l_q *defines an element of the dual of* l_p, l_p^* *in this way, and we have*

$$\|\phi\|_d = \|y\|_q = \begin{cases} \left(\sum_{t=0}^{\infty} |y_t|^q\right)^{\frac{1}{q}} & \text{if } 1 < p < \infty \\ \sup_t |y_t| & \text{if } p = 1 \end{cases}$$

Let's first understand what the theorem gives us. Take any space l_p (note that the theorem does NOT make any statements about l_∞). Then the theorem states that its dual is l_q. The first part of the theorem states that $l_q \subseteq l_p^*$. Take any element $\phi \in l_p^*$. Then there exists $y \in l_q$ such that ϕ is representable by y. In this sense $\phi \in l_q$. The second part states that any $y \in l_q$ defines a functional ϕ on l_p by (38). Given its definition, ϕ is obviously continuous and hence bounded. Finally the theorem assures that the norm of the functional ϕ associated with y is indeed the norm associated with l_q. Hence $l_p^* \subseteq l_q$.

As a result of the theorem, whenever we deal with l_p, $p \in [1, \infty)$ as commodity space we can restrict attention to price systems that can be represented by a vector $p = (p_0, p_1, \ldots p_t, \ldots)$ and hence have a straightforward economic interpretation: p_t is the price of the good at period t and the cost of a consumption bundle x is just the sum of the cost of all its components.

For reasons that will become clearer later the most interesting commodity space for infinitely lived economies, however, is l_∞. And for this commodity space the previous theorem does not make any statements. It would suggest that the dual of l_∞ is l_1, but this is not quite correct, as the next result shows.

Proposition 59 *The dual of* l_∞ *contains* l_1. *There are* $\phi \in l_\infty^*$ *that are not representable by an element* $y \in l_1$

Proof. For the first part for any $y \in l_1$ define $\phi : l_\infty \to \mathbf{R}$ by

$$\phi(x) = \sum_{t=0}^{\infty} x_t y_t$$

We need to show that ϕ is linear and continuous. Linearity is obvious. For continuity we need to show that for any sequence $\{x^n\} \in l_\infty$ and $x \in l_\infty$, $\|x^n - x\| = \sup_t |x_t^n - x_t| \to 0$ implies $|\phi(x^n) - \phi(x)| \to 0$. Since $y \in l_1$ there exists M such that $\sum_{t=0}^{\infty} |y_t| < M$. Since $\sup_t |x_t^n - x_t| \to 0$, for all $\delta > 0$ there exists $N(\delta)$ such that fro all $n > N(\delta)$ we have $\sup_t |x_t^n - x_t| < \delta$. But then for any $\varepsilon > 0$, taking $\delta(\varepsilon) = \frac{\varepsilon}{2M}$ and $N(\varepsilon) = N(\delta(\varepsilon))$, for all $n > N(\varepsilon)$

$$\begin{aligned}
|\phi(x^n) - \phi(x)| &= \left| \sum_{t=0}^{\infty} x_t^n y_t - \sum_{t=0}^{\infty} x_t y_t \right| \\
&\leq \sum_{t=0}^{\infty} |y_t(x_t^n - x_t)| \\
&\leq \sum_{t=0}^{\infty} |y_t| \cdot |x_t^n - x_t| \\
&\leq M\delta(e) = \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

The second part we prove via a counter example after we have proved the second welfare theorem. ■

The second part of the proposition is somewhat discouraging in that it asserts that, when dealing with l_∞ as commodity space we may require a price system that does not have a natural economic interpretation. It is true that there is a subspace of l_∞ for which l_1 is its dual. Define the space c_0 (with associated sup-norm) as

$$c_0 = \{x \in l_\infty : \lim_{t \to \infty} x_t = 0\}$$

We can prove that l_1 is the dual of c_0. Since $c_0 \subseteq l_\infty$ and $l_1 \subseteq l_\infty^*$, obviously $l_1 \subseteq c_0^*$. It remains to show that any $\phi \in c_0^*$ can be represented by a $y \in l_1$. [TO BE COMPLETED]

7.3 Definition of Competitive Equilibrium

Corresponding to our two notions of an economy and a private ownership economy we have two definitions of competitive equilibrium that differ in their specification of the individual budget constraints.

Definition 60 *A competitive equilibrium is an allocation* $[(x_i^0)_{i \in I}, (y_j^0)_{j \in J}]$ *and a continuous linear functional* $\phi : S \to \mathbf{R}$ *such that*

1. *for all* $i \in I$, x_i^0 *solves* $\max u_i(x)$ *subject to* $x \in X_i$ *and* $\phi(x) \leq \phi(x_i^0)$

2. *for all* $j \in J$, y_j^0 *solves* $\max \phi(y)$ *subject to* $y \in Y_j$

3. $\sum_{i \in I} x_i^0 = \sum_{j \in J} y_j^0$

In this definition we have obviously ignored ownership of firms. If, however, all Y_j are convex cones, the technologies exhibit constant returns to scale, profits are zero in equilibrium and this definition of equilibrium is equivalent to the definition of equilibrium for a private ownership economy (under appropriate assumptions on preferences such as local nonsatiation). Note that condition 1. is equivalent to requiring that for all $i \in I$, $x \in X_i$ and $\phi(x) \leq \phi(x_i^0)$ implies $u_i(x) \leq u_i(x_i^0)$ which states that all bundles that are cheaper than x_i^0 must not yield higher utility. Again note that we made no reference to the value of an individuals' endowment or firm ownership.

Definition 61 *A competitive equilibrium for a private ownership economy is an allocation* $[(x_i^0)_{i \in I}, (y_j^0)_{j \in J}]$ *and a continuous linear functional* $\phi : S \to \mathbf{R}$ *such that*

1. *for all* $i \in I$, x_i^0 *solves* $\max u_i(x)$ *subject to* $x \in X_i$ *and* $\phi(x) \leq \sum_{j \in J} \theta_{ij} \phi(y_j^0)$

2. *for all* $j \in J$, y_j^0 *solves* $\max \phi(y)$ *subject to* $y \in Y_j$

3. $\sum_{i \in I} x_i^0 = \sum_{j \in J} y_j^0$

We can interpret $\sum_{j \in J} \theta_{ij} \phi(y_j^0)$ as the value of the ownership that household i holds to all the firms of the economy.

7.3.1 The Neoclassical Growth Model in Arrow-Debreu Language

Let us look at the neoclassical growth model presented in Section 2. We will adopt the notation so that it fits into our general discussion. Remember that in the economy the representative household owned the capital stock and the representative firm, supplied capital and labor services and bought final output from the firm. A helpful exercise would be to repeat this exercise under the assumption that the firm owns the capital stock. The household had unit endowment of time and initial endowment of $\bar{k}_0$ of the capital stock. To make our exercise more interesting we assume that the household values consumption and leisure according to instantaneous utility function $U(c, l)$, where c is consumption and l is leisure. The technology is described by $y = F(k, n)$ where F exhibits constant returns to scale. For further details refer to Section 2. Let us represent this economy in Arrow-Debreu language.

- $I = J = 1, \theta_{ij} = 1$
- Commodity Space S: since three goods are traded in each period (final output, labor and capital services), time is discrete and extends to infinity,

a natural choice is $S = l_\infty^3 = l_\infty \times l_\infty \times l_\infty$. That is, S consists of all three-dimensional infinite sequences that are bounded in the sup-norm, or

$$S = \{s = (s^1, s^2, s^3) = \{(s_t^1, s_t^2, s_t^3)\}_{t=0}^\infty : s_t^i \in \mathbf{R}, \sup_t \max_i \left|s_t^i\right| < \infty\}$$

Obviously S, together with the sup-norm, is a (real) normed vector space. We use the convention that the first component of s denotes the output good (and hence is required to be positive), whereas the second and third components denote labor and capital services, respectively. Again following the convention these inputs are required to be negative.

- Consumption Set X :

  $$X = \{\{x_t^1, x_t^2, x_t^3\} \in S : x_0^3 \geq -\bar{k}_0, -1 \leq x_t^2 \leq 0, x_t^3 \leq 0, x_t^1 \geq 0, x_t^1 - (1-\delta)x_t^3 + x_{t+1}^3 \geq 0 \text{ for all } t\}$$

  We do not distinguish between capital and capital services here; this can be done by adding extra notation and is an optional homework. The constraints indicate that the household cannot provide more capital in the first period than the initial endowment, can't provide more than one unit of labor in each period, holds nonnegative capital stock and is required to have nonnegative consumption. Evidently $X \subseteq S$.

- Utility function $u : X \to \mathbf{R}$ is defined by

  $$u(x) = \sum_{t=0}^{\infty} \beta^t U(x_t^1 - (1-\delta)x_t^3 + x_{t+1}^3, 1 + x_t^2)$$

  Again remember the convention than labor and capital (as inputs) are negative.

- Aggregate Production Set Y :

  $$Y = \{\{y_t^1, y_t^2, y_t^3\} \in S : y_t^1 \geq 0, y_t^2 \leq 0, y_t^3 \leq 0, y_t^1 = F(-y_t^3, -y_t^2) \text{ for all } t\}$$

  Note that the aggregate production set reflects the technological constraints in the economy. It does not contain any constraints that have to do with limited supply of factors, in particular $-1 \leq y_t^2$ is not imposed.

- An allocation is $[x, y]$ with $x, y \in S$. A feasible allocation is an allocation such that $x \in X, y \in Y$ and $x = y$. An allocation is Pareto optimal is it is feasible and if there is no other feasible allocation $[x^*, y^*]$ such that $u(x^*) > u(x)$.

- A price system ϕ is a continuous linear functional $\phi : S \to \mathbf{R}$. If ϕ has inner product representation, we represent it by $p = (p^1, p^2, p^3) = \{(p_t^1, p_t^2, p_t^3)\}_{t=0}^\infty$.

- A competitive equilibrium for this private ownership economy is an allocation $[x^*, y^*]$ and a continuous linear functional such that

  1. y^* maximizes $\phi(y)$ subject to $y \in Y$
  2. x^* maximizes $u(x)$ subject to $x \in X$ and $\phi(x) \leq \phi(y^*)$
  3. $x^* = y^*$

Note that with constant returns to scale $\phi(y^*) = 0$. With inner product representation of the price system the budget constraint hence becomes

$$\phi(x) = p \cdot x = \sum_{t=0}^{\infty} \sum_{i=1}^{3} p_t^i x_t^i \leq 0$$

Remembering our sign convention for inputs and mapping $p_t^1 = p_t$, $p_t^2 = p_t w_t$, $p_t^3 = p_t r_t$ we obtain the same budget constraint as in Section 2.

7.3.2 A Pure Exchange Economy in Arrow-Debreu Language

Suppose there are I individuals that live forever. There is one nonstorable consumption good in each period. Individuals order consumption allocations according to

$$u_i(c_i) = \sum_{t=0}^{\infty} \beta_i^t U(c_t^i)$$

They have deterministic endowment streams $e^i = \{e_t^i\}_{t=0}^{\infty}$. Trade takes place at period 0. The standard definition of a competitive (Arrow-Debreu) equilibrium would go like this:

Definition 62 *A competitive equilibrium are prices* $\{p_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i\in I}$ *such that*

1. *Given* $\{p_t\}_{t=0}^{\infty}$, *for all* $i \in I$, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ *solves* $\max_{c^i \geq 0} u_i(c_i)$ *subject to*

$$\sum_{t=0}^{\infty} p_t(c_t^i - e_t^i) \leq 0$$

2.

$$\sum_{i\in I} c_t^i = \sum_{i \in I} e_t^i \text{ for all } t$$

We briefly want to demonstrate that we can easily write this economy in our formal language. What goes on is that the household sells his endowment of the consumption good to the market and buys consumption goods from the market. So even though there is a single good in each period we find it useful to have two commodities in each period. We also introduce an artificial technology that transforms one unit of the endowment in period t into one unit of the consumption good at period t. There is a single representative firm that operates this technology and each consumer owns share θ_i of the firm, with $\sum_{i\in I}\theta_i = 1$. We then have the following representation of this economy

- $S = l_\infty^2$. We use the convention that the first good is the consumption good to be consumed, the second good is the endowment to be sold as input by consumers. Again we use the convention that final output is positive, inputs are negative.

- $X_i = \{x \in S : x_t^1 \geq 0, -e_t^i \leq x_t^2 \leq 0\}$

- $u_i : X_i \to \mathbf{R}$ defined by

$$u_i(x) = \sum_{t=0}^{\infty} \beta_i^t U(x_t^1)$$

- Aggregate production set

$$Y = \{y \in S : y_t^1 \geq 0, y_t^2 \leq 0, y_t^1 = -y_t^2\}$$

- Allocations, feasible allocations and Pareto efficient allocations are defined as before.

- A price system ϕ is a continuous linear functional $\phi : S \to \mathbf{R}$. If ϕ has inner product representation, we represent it by $p = (p^1, p^2) = \{(p_t^1, p_t^2)\}_{t=0}^{\infty}$.

- A competitive equilibrium $[(x^{i*})_{i\in I}, y, \phi]$ for this private ownership economy defined as before.

- Note that with constant returns to scale in equilibrium we have $\phi(y^*) = 0$. With inner product representation of the price system in equilibrium also $p_t^1 = p_t^2 = p_t$. The budget constraint hence becomes

$$\phi(x) = p \cdot x = \sum_{t=0}^{\infty}\sum_{i=1}^{2} p_t^i x_t^i \leq 0$$

  Obviously (as long as $p_t > 0$ for all t) the consumer will choose $x_t^{i2} = -e_t^i$, i.e. sell all his endowment. The budget constraint then takes the familiar form

$$\sum_{t=0}^{\infty} p_t(c_t^i - e_t^i) \leq 0$$

The purpose of this exercise was to demonstrate that, although in the remaining part of the course we will describe the economy and define an equilibrium in the first way, whenever we desire to prove the welfare theorems we can represent any pure exchange economy easily in our formal language and use the machinery developed in this section (if applicable).

7.4 The First Welfare Theorem

The first welfare theorem states that every competitive equilibrium allocation is Pareto optimal. The only assumption that is required is that people's preferences be locally nonsatiated. The proof of the theorem is unchanged from the one you should be familiar with from micro last quarter

Theorem 63 *Suppose that for all i, all $x \in X_i$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X_i converging to x with $u(x_n) > u(x)$ for all n (local nonsatiation). If an allocation $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ and a continuous linear functional ϕ constitute a competitive equilibrium, then the allocation $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is Pareto optimal.*

Proof. The proof is by contradiction. Suppose $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$, ϕ is a competitive equilibrium.

Step 1: We show that for all i, all $x \in X_i$, $u(x) \geq u(x_i^0)$ implies $\phi(x) \geq \phi(x_i^0)$. Suppose not, i.e. suppose there exists i and $x \in X_i$ with $u(x) \geq u(x_i^0)$ and $\phi(x) < \phi(x_i^0)$. Let $\{x_n\}$ in X_i be a sequence converging to x with $u(x_n) > u(x)$ for all n. Such a sequence exists by our local nonsatiation assumption. By continuity of ϕ there exists an n such that $u(x_n) > u(x) \geq u(x_i^0)$ and $\phi(x_n) < \phi(x_i^0)$, violating the fact that x_i^0 is part of a competitive equilibrium.

Step 2: For all i, all $x \in X_i$, $u(x) > u(x_i^0)$ implies $\phi(x) > \phi(x_i^0)$. This follows directly from the fact that x_i^0 is part of a competitive equilibrium.

Step 3: Now suppose $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is not Pareto optimal. Then there exists another feasible allocation $[(x_i^*)_{i\in I}, (y_j^*)_{j\in J}]$ such that $u(x_i^*) \geq u(x_i^0)$ for all i and with strict inequality for some i. Since $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is a competitive equilibrium allocation, by step 1 and 2 we have

$$\phi(x_i^*) \geq \phi(x_i^0)$$

for all i, with strict inequality for some i. Summing up over all individuals yields

$$\sum_{i\in I} \phi(x_i^*) > \sum_{i\in I} \phi(x_i^0) < \infty$$

The last inequality comes from the fact that the set of people I is finite and that for all i, $\phi(x_i^0)$ is finite (otherwise the consumer maximization problem has no solution). By linearity of ϕ we have

$$\phi\left(\sum_{i\in I} x_i^*\right) = \sum_{i\in I} \phi(x_i^*) > \sum_{i\in I} \phi(x_i^0) = \phi\left(\sum_{i\in I} x_i^0\right)$$

Since both allocations are feasible we have that

$$\begin{aligned} \sum_{i\in I} x_i^0 &= \sum_{j\in J} y_j^0 \\ \sum_{i\in I} x_i^* &= \sum_{j\in J} y_j^* \end{aligned}$$

and hence

$$\phi\left(\sum_{j\in J} y_j^*\right) > \phi\left(\sum_{j\in J} y_j^0\right)$$

Again by linearity of ϕ

$$\sum_{j\in J}\phi(y_j^*) > \sum_{j\in J}\phi(y_j^0)$$

and hence for at least one $j \in J$, $\phi(y_j^*) > \phi(y_j^0)$. But $y_j^* \in Y_j$ and we obtain a contradiction to the hypothesis that $[(x_i^0)_{i\in I}, (y_j^0)_{j\in J}]$ is a competitive equilibrium allocation. ■

Several remarks are in order. It is crucial for the proof that the set of individuals is finite, as will be seen in our discussion of overlapping generations economies. Also our equilibrium definition seems odd as it makes no reference to endowments or ownership in the budget constraint. For the preceding theorem, however, this is not a shortcoming. Since we start with a competitive equilibrium we know the value of each individual's consumption allocation. By local nonsatiation each consumer exhausts her budget and hence we implicitly know each individual's income (the value of endowments and firm ownership, if specified in a private ownership economy).

7.5 The Second Welfare Theorem

The second welfare theorem provides a converse to the first welfare theorem. Under suitable assumptions it states that for any Pareto-optimal allocation there exists a price system such that the allocation together with the price system form a competitive equilibrium. It may at first be surprising that the second welfare theorem requires much more stringent assumptions than the first welfare theorem. Remember, however, that in the first welfare theorem we start with a competitive equilibrium whereas in the proof of the second welfare we have to carry out an existence proof. Comparing the assumptions of the second welfare theorem with those of existence theorems makes clear the intimate relation between them.

As in micro we will use a separating hyperplane theorem to establish the existence of a price system that decentralizes a given allocation $[x, y]$. The price system is nothing else than a hyperplane that separates the aggregate production set from the set of consumption allocations that are jointly preferred by all consumers. Figure 3 illustrates this general principle.In lieu of Figure 3 it is not surprising that several convexity assumptions have to be made to prove the second welfare theorem. We will come back to this when we discuss each specific assumption. First we state the separating hyperplane that we will use for our proof. Obviously we can't use the standard theorems commonly used in micro[34] since our commodity space in a general real vector space (possibly infinite dimensional).

[34] See MasColell et al., p. 948. This theorem is usually attributed to Minkowski.

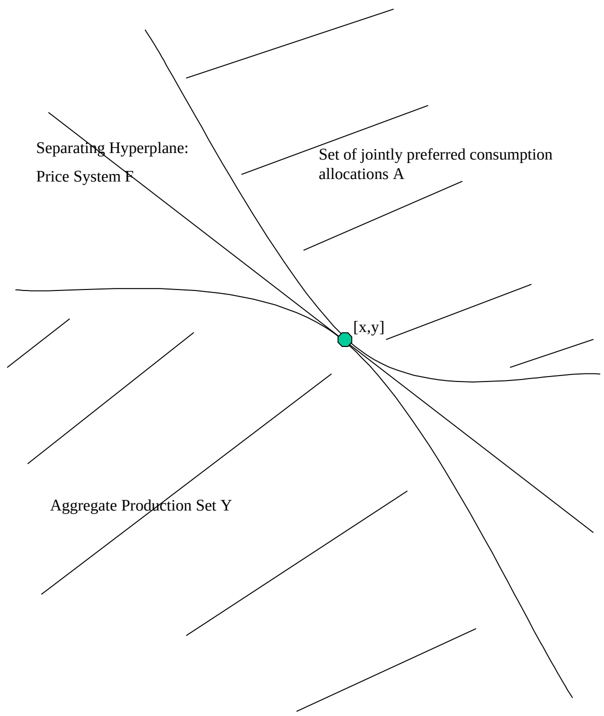
Separating Hyperplane:
Price System F
Set of jointly preferred consumption
allocations A
[x,y]
Aggregate Production Set Y

Figure 6:

We will apply the geometric form of the Hahn-Banach theorem. For this we need the following definition

Definition 64 *Let S be a normed real vector space with norm $\|\|_S$. Define by*

$$b(x,\varepsilon) = \{s \in S : \|x-s\|_S < \varepsilon\}$$

the open ball of radius ε around x. The interior of a set $A \subseteq S$, $\mathring{A}$ is defined to be

$$\mathring{A} = \{x \in A : \exists \varepsilon > 0 \text{ with } b(x,\varepsilon) \subseteq A\}$$

Hence the interior of a set A consists of all the points in A for which we can find a open ball (no matter how small) around the point that lies entirely in A. We then have the following

Theorem 65 *(Geometric Form of the Hahn-Banach Theorem): Let $A, Y \subset S$ be convex sets and assume that*

$$\begin{aligned} \text{either } Y \text{ has an interior point and } A \cap \mathring{Y} &= \emptyset \\ \text{or } S \text{ is finite dimensional and } A \cap Y &= \emptyset \end{aligned}$$

Then there exists a continuous linear functional ϕ, not identically zero on S, and a constant c such that

$$\phi(y) \leq c \leq \phi(x) \text{ for all } x \in A \text{ and all } y \in Y$$

For the proof of the Hahn-Banach theorem in its several forms see Luenberger (1969), p. 111 and p. 133. For the case that S is finite dimensional this theorem is rather intuitive in light of Figure 3. But since we are interested in commodity spaces with infinite dimensions (typically $S = l_p$, for $p \in [1,\infty]$), we usually have to prove that the aggregate production set Y has an interior point in order to apply the Hahn-Banach theorem. We will two things now: a) prove by example that the requirement of an interior point is an assumption that cannot be dispensed with if S is not finite dimensional b) show that this assumption de facto rules out using $S = l_p$, for $p \in [1,\infty)$, as commodity space when one wants to apply the second welfare theorem.

For the first part consider the following

Example 66 *Consider as commodity space*

$$S = \{\{x_t\}_{t=0}^{\infty} : x_t \in \mathbf{R} \text{ for all } t,\ \|x\|_S = \sum_{t=0}^{\infty} \beta^t |x_t| < \infty\}$$

for some $\beta \in (0,1)$. Let $A = \{\theta\}$ and

$$Y = \{x \in S : |x_t| \leq 1 \text{ for all } t\}$$

Obviously $A, B \subset S$ are convex sets. In some sense $\theta = (0, 0, \ldots, 0, \ldots)$ lies in the middle of Y, but it does not lie in the interior of Y. Suppose it did, then there exists $\varepsilon > 0$ such that for all $x \in S$ such that

$$\|x - \theta\|_S = \sum_{t=0}^{\infty} \beta^t |x_t| < \varepsilon$$

we have $x \in Y$. But for any $\varepsilon > 0$, define $t(\varepsilon) = \frac{\ln(\frac{\varepsilon}{2})}{\ln(\beta)} + 1$. Then $x = (0, 0, \ldots, x_{t(\varepsilon)} = 2, 0, \ldots) \notin Y$ satisfies $\sum_{t=0}^{\infty} \beta^t |x_t| = 2\beta^{t(\varepsilon)} < \varepsilon$. Since this is true for all $\varepsilon > 0$, this shows that θ is not in the interior of Y, or $A \cap \overset{\circ}{Y} = \emptyset$. A very similar argument shows that no $s \in S$ is in the interior of Y, i.e. $\overset{\circ}{Y} = \emptyset$. Hence the only hypothesis for the Hahn-Banach theorem that fails is that Y has an interior point. We now show that the conclusion of the theorem fails. Suppose, to the contrary, that there exists a continuous linear functional ϕ on S with $\phi(s) \neq 0$ for some $\bar{s} \in S$ and

$$\phi(y) \leq c \leq \phi(\theta) \text{ for all } y \in Y$$

Obviously $\phi(\theta) = \phi(0 \cdot \bar{s}) = 0$ by linearity of ϕ. Hence it follows that for all $y \in Y$, $\phi(y) \leq 0$. Now suppose there exists $\bar{y} \in Y$ such that $\phi(\bar{y}) < 0$. But since $-\bar{y} \in Y$, by linearity $\phi(-\bar{y}) = -\phi(\bar{y}) > 0$ a contradiction. Hence $\phi(y) = 0$ for all $y \in Y$. From this it follows that $\phi(s) = 0$ for all $s \in S$ (why?), contradicting the conclusion of the theorem.

As we will see in the proof of the second welfare theorem, to apply the Hahn-Banach theorem we have to assure that the aggregate production set has nonempty interior. The aggregate production set in many application will be (a subset) of the positive orthant of the commodity space. The problem with taking l_p, $p \in [1, \infty)$ as the commodity space is that, as the next proposition shows, the positive orthant

$$l_p^+ = \{x \in l_p : x_t \geq 0 \text{ for all } t\}$$

has empty interior. The good thing about l_∞ is that is has a nonempty interior. This justifies why we usually use it (or its k-fold product space) as commodity space.

Proposition 67 *The positive orthant of l_p, $p \in [0, \infty)$ has an empty interior. The positive orthant of l_∞ has nonempty interior.*

Proof. For the first part suppose there exists $x \in l_p^+$ and $\varepsilon > 0$ such that $b(x, \varepsilon) \subseteq l_p^+$. Since $x \in l_p$, $x_t \to 0$, i.e. $x_t < \frac{\varepsilon}{2}$ for all $t \geq T(\varepsilon)$. Take any $\tau > T(\varepsilon)$ and define z as

$$z_t = \begin{cases} x_t \text{ if } t \neq \tau \\ x_t - \frac{\varepsilon}{2} \text{ if } t = \tau \end{cases}$$

Evidently $z_\tau < 0$ and hence $z \notin l_p^+$. But since

$$\|x - z\|_p = \left(\sum_{t=0}^{\infty} |x_t - z_t|^p \right)^{\frac{1}{p}} = |x_\tau - z_\tau| = \frac{\varepsilon}{2} < \varepsilon$$

we have $z \in b(x, \varepsilon)$, a contradiction. Hence the interior of l_p^+ is empty, the Hahn-Banach theorem doesn't apply and we can't use it to prove the second welfare theorem.

For the second part it suffices to construct an interior point of l_∞^+. Take $x = (1, 1, \ldots, 1, \ldots)$ and $\varepsilon = \frac{1}{2}$. We want to show that $b(x, \varepsilon) \subseteq l_\infty^+$. Take any $z \in b(x, \varepsilon)$. Clearly $z_t \geq \frac{1}{2} \geq 0$. Furthermore

$$\sup_t |z_t| \leq 1\frac{1}{2} < \infty$$

Hence $z \in l_\infty^+$. ■

Now let us proceed with the statement and the proof of the second welfare theorem. We need the following assumptions

1. For each $i \in I$, X_i is convex.
2. For each $i \in I$, if $x, x' \in X_i$ and $u_i(x) > u_i(x')$, then for all $\lambda \in (0, 1)$
$$u_i(\lambda x + (1 - \lambda)x') > u_i(x')$$
3. For each $i \in I$, u_i is continuous.
4. The aggregate production set Y is convex
5. Either Y has an interior point or S is finite-dimensional.

Note that the second assumption is sometimes referred to as strict quasi-concavity[35] of the utility functions. It implies that the upper contour sets

$$A_x^i = \{z \in X_i : u_i(z) \geq u_i(x)\}$$

are convex, for all i, all $x \in X_i$. Without the convexity assumption 1. assumption 2 would not be well-defined as without convex X_i, $\lambda x + (1-\lambda)x' \notin X_i$ is possible, in which case $u_i(\lambda x + (1-\lambda)x')$ is not well-defined. I mention this since otherwise 1. is not needed for the following theorem. Also note that it is assumption 5 that has no counterpart to the theorem in finite dimensions. It only is required to use the appropriate separating hyperplane theorem in the proof. With these assumptions we can state the second welfare theorem

Theorem 68 *Let* $[(x_i^0), (y_j^0)]$ *be a Pareto optimal allocation and assume that for some* $h \in I$ *there is a* $\hat{x}_h \in X_h$ *with* $u_h(\hat{x}_h) > u_h(x_h^0)$. *Then there exists a continuous linear functional* $\phi : S \to \mathbf{R}$, *not identically zero on* S, *such that*

[35] To me it seems that quasi-concavity is enough for the theorem to hold as quasi-concavity is equivalent to convex upper contour sets which all one needs in the proof.

1. *for all* $j \in J$, $y_j^0 \in \arg\max_{y \in Y_j} \phi(y)$

2. *for all* $i \in I$ *and all* $x \in X_i$, $u_i(x) \geq u_i(x_i^0)$ *implies* $\phi(x) \geq \phi(x_i^0)$

Several comments are in order. The theorem states that (under the assumptions of the theorem) any Pareto optimal allocation can be supported by a price system as a quasi-equilibrium. By definition of Pareto optimality the allocation is feasible and hence satisfies resource balance. The theorem also guarantees profit maximization of firms. For consumers, however, it only guarantees that x_i^0 minimizes the cost of attaining utility $u_i(x_i^0)$, but not utility maximization among the bundles that cost no more than $\phi(x_i^0)$, as would be required by a competitive equilibrium. You also may be used to a version of this theorem that shows that a Pareto optimal allocation can be made into an equilibrium with transfers. Since here we haven't defined ownership and in the equilibrium definition make no reference to the value of endowments or firm ownership (i.e. do NOT require the budget constraint to hold), we can abstract from transfers, too. The proof of the theorem is similar to the one for finite dimensional commodity spaces.

Proof. Let $[(x_i^0), (y_j^0)]$ be a Pareto optimal allocation and $A^i_{x_i^0}$ be the upper contour sets (as defined above) with respect to x_i^0, for all $i \in I$. Also let $\mathring{A}^i_{x_i^0}$ to be the interior of $A^i_{x_i^0}$, i.e.

$$\mathring{A}^i_{x_i^0} = \{z \in X_i : u_i(z) > u_i(x_i^0)\}$$

By assumption 2. the $A^i_{x_i^0}$ are convex and hence $\mathring{A}^i_{x_i^0}$ is convex. Furthermore $x_i^0 \in A^i_{x_i^0}$, so the $A^i_{x_i^0}$ are nonempty. By one of the hypotheses of the theorem there is some $h \in I$ there is a $\hat{x}_h \in X_h$ with $u_h(\hat{x}_h) > u_h(x_h^0)$. For that h, $\mathring{A}^h_{x_h^0}$ is nonempty. Define

$$A = \mathring{A}^h_{x_h^0} + \sum_{i \neq h} A^i_{x_i^0}$$

A is the set of all aggregate consumption bundles that can be split in such a way as to give every agent at least as much utility and agent h strictly more utility than the Pareto optimal allocation $[(x_i^0), (y_j^0)]$. As A is the sum of nonempty convex sets, so is A. Obviously $A \subset S$. By assumption Y is convex. Since $[(x_i^0), (y_j^0)]$ is a Pareto optimal allocation $A \cap Y = \emptyset$. Otherwise there is an aggregate consumption bundle $x^* \in A \cap Y$ that can be produced (as $x^* \in Y$) and Pareto dominates x^0 (as $x^* \in A$), contradicting Pareto optimality of $[(x_i^0), (y_j^0)]$. With assumption 5. we have all the assumptions we need to apply the Hahn-Banach theorem. Hence there exists a continuous linear functional ϕ on S, not identically zero, and a number c such that

$$\phi(y) \leq c \leq \phi(x) \text{ for all } x \in A, \text{ all } y \in Y$$

It remains to be shown that $[(x_i^0), (y_j^0)]$ together with ϕ satisfy conclusions 1 and 2, i.e. constitute a quasi-equilibrium.

First note that the closure of A is $\bar{A} = \sum_{i \in I} A^i_{x_i^0}$ since by continuity of u_h (assumption 3.) the closure of $\mathring{A}^h_{x_h^0}$ is $A^h_{x_h^0}$. Therefore, since ϕ is continuous, $c \leq \phi(x)$ for all $x \in \bar{A} = \sum_{i \in I} A^i_{x_i^0}$.

Second, note that, since $[(x_i^0), (y_j^0)]$ is Pareto optimal, it is feasible and hence $y^0 \in Y$

$$x^0 = \sum_{i \in I} x_i^0 = \sum_{j \in J} y_j^0 = y^0$$

Obviously $x^0 \in \bar{A}$. Therefore $\phi(x^0) = \phi(y^0) \leq c \leq \phi(x^0)$ which implies $\phi(x^0) = \phi(y^0) = c$.

To show conclusion 1 fix $j \in J$ and suppose there exists $\tilde{y}_j \in Y_j$ such that $\phi(\tilde{y}_j) > \phi(y_j^0)$. For $k \neq j$ define $\tilde{y}_k = y_k^0$. Obviously $\tilde{y} = \sum_j \tilde{y}_j \in Y$ and $\phi(\tilde{y}) > \phi(y^0) = c$, a contradiction to the fact that $\phi(y) \leq c$ for all $y \in Y$. Therefore y_j^0 maximizes $\phi(z)$ subject to $z \in Y_j$, for all $j \in J$.

To show conclusion 2 fix $i \in I$ and suppose there exists $\tilde{x}_i \in X_i$ with $u_i(\tilde{x}_i) \geq u_i(x_i^0)$ and $\phi(\tilde{x}_i) < \phi(x_i^0)$. For $l \neq i$ define $\tilde{x}_l = x_l^0$. Obviously $\tilde{x} = \sum_i \tilde{x}_i \in \bar{A}$ and $\phi(\tilde{x}) < \phi(x^0) = c$, a contradiction to the fact that $\phi(x) \geq c$ for all $x \in \bar{A}$. Therefore x_i^0 minimizes $\phi(z)$ subject to $u_i(z) \geq u_i(x_i^0), z \in X_i$. ■

We now want to provide a condition that assures that the quasi-equilibrium in the previous theorem is in fact a competitive equilibrium, i.e. is not only cost minimizing for the households, but also utility maximizing. This is done in the following

Remark 69 *Let the hypotheses of the second welfare theorem be satisfied and let ϕ be a continuous linear functional that together with $[(x_i^0), (y_j^0)]$ satisfies the conclusions of the second welfare theorem. Also suppose that for all $i \in I$ there exists $x_i' \in X_i$ such that*

$$\phi(x_i') < \phi(x_i^0)$$

Then $[(x_i^0), (y_j^0), \phi]$ constitutes a competitive equilibrium

Note that, in order to verify the additional condition -the existence of a cheaper point in the consumption set for each $i \in I$- we need a candidate price system ϕ that already passed the test of the second welfare theorem. It is not, as the assumptions for the second welfare theorem, an assumptions on the fundamentals of the economy alone.

Proof. We need to prove that for all $i \in I$, all $x \in X_i$, $\phi(x) \leq \phi(x_i^0)$ implies $u_i(x) \leq u_i(x_i^0)$. Pick an arbitrary $i \in I$, $x \in X_i$ satisfying $\phi(x) \leq \phi(x_i^0)$. Define

$$x_\lambda = \lambda x_i' + (1 - \lambda)x \text{ for all } \lambda \in (0, 1)$$

Since by assumption $\phi(x_i') < \phi(x_i^0)$ and $\phi(x) \leq \phi(x_i^0)$ we have by linearity of ϕ

$$\phi(x_\lambda) = \lambda\phi(x_i') + (1 - \lambda)\,\phi(x) < \phi(x_i^0) \text{ for all } \lambda \in (0, 1)$$

Since x_0^i by assumption is part of a quasi-equilibrium and (by convexity of X_i we have $x_\lambda \in X_i$), $u_i(x_\lambda) \geq u_i(x_i^0)$ implies $\phi(x_\lambda) \geq \phi(x_i^0)$, or by contraposition $\phi(x_\lambda) < \phi(x_i^0)$ implies $u_i(x_\lambda) < u_i(x_i^0)$ for all $\lambda \in (0, 1)$. But then by continuity of u_i we have $u_i(x) = \lim_{\lambda \to 0} u_i(x_\lambda) \leq u_i(x_i^0)$ as desired. ■

As shown by an example in Stokey et al. the assumption on the existence of a cheaper point cannot be dispensed with when wanting to make sure that a quasi-equilibrium is in fact a competitive equilibrium. In Figure 4 we draw the Edgeworth box of a pure exchange economy. Consumer B's consumption set is the entire positive orthant, whereas consumer A's consumption set is the are above the line marked by $-p$, as indicated by the broken lines. Both consumption sets are convex, the upper contour sets are convex and close as for standard utility functions satisfying assumptions 2. and 3. Point E clearly represents a Pareto optimal allocation (since at E consumer B's utility is globally maximized subject to the allocation being feasible). Furthermore E represents a quasi-equilibrium, since at prices p both consumers minimize costs subject to attaining at least as much utility as with allocation E. However, at prices p (obviously the only candidate for supporting E as competitive equilibrium since tangent to consumer B's indifference curve through E) agent A obtains higher utility at allocation E' with the same cost as with E, hence $[E, p]$ is not a competitive equilibrium. The remark fails because at candidate prices p there is no consumption allocation for A that is feasible (in X_A) and cheaper. This demonstrates that the cheaper-point assumption cannot be dispensed with in the remark. This concludes the discussion of the second welfare theorem.

The last thing we want to do in this section is to demonstrate that our choice of l_∞ as commodity space is not without problems either. We argued earlier that l_p, $p \in [1, \infty)$ is not an attractive alternative. Now we use the second welfare theorem to show that for certain economies the price system needed (whose existence is guaranteed by the theorem) need not lie in l_1, i.e. does not have a representation as a vector $p = (p_0, p_1, \ldots, p_t, \ldots)$. This is bad in the sense that then the price system we get from the theorem does not have a natural economic interpretation. After presenting such a pathological example we will briefly discuss possible remedies.

Example 70 *Let $S = l_\infty$. There is a single consumer and a single firm. The aggregate production set is given by*

$$Y = \{y \in S : 0 \leq y_t \leq 1 + \frac{1}{t}, \text{ for all } t\}$$

The consumption set is given by

$$X = \{x \in S : x_t \geq 0 \text{ for all } t\}$$

The utility function $u : X \to \mathbf{R}$ is

$$u(x) = \inf_t x_t$$

[TO BE COMPLETED]

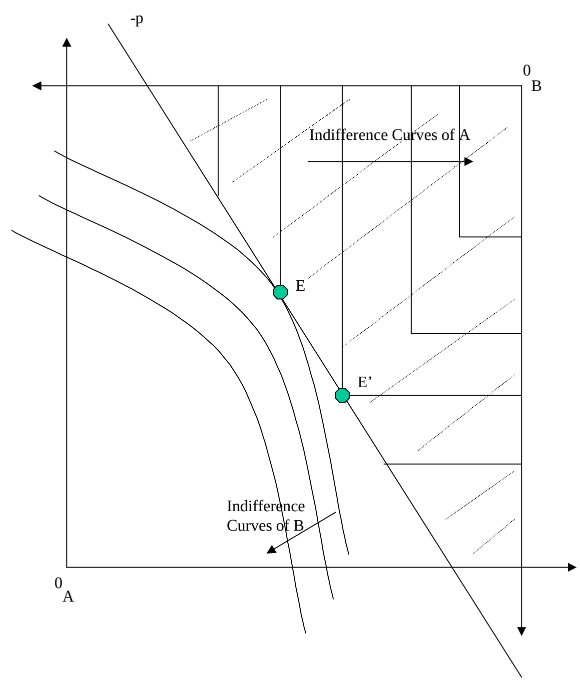

Figure 7:

7.6 Type Identical Allocations

[TO BE COMPLETED]