8 The Overlapping Generations Model

This this section we will discuss the second major workhorse model of modern macroeconomics, the Overlapping Generations (OLG) model, due to Allais (1947), Samuelson (1958) and Diamond (1965). The structure of this section will be as follows: we will first present a basic pure exchange version of the OLG model, show how to analyze it and contrast its properties with those of a pure exchange economy with infinitely lived agents. The basic differences are that in the OLG model

- competitive equilibria may be Pareto suboptimal
- (outside) money may have positive value
- there may exist a continuum of equilibria
- the core of the OLG economy may be empty

We will demonstrate the first three of these properties in detail via examples and leave the fourth property to further reading. We will then discuss the Ricardian Equivalence hypothesis (the notion that, given a stream of government spending the financing method of the government -taxes or budget deficits- does not influence macroeconomic aggregates) for both the infinitely lived agent model as well as the OLG model. Finally we will introduce production into the OLG model to discuss the notion of dynamic inefficiency. The first part of this section will be based on Kehoe (1989), Geanakoplos (1989), the second section on Barro (1974) and the third section on Diamond (1965). Other good sources of information include Blanchard and Fischer (1989), chapter 3, Sargent and Ljungquist, chapter 8 and Azariadis, chapter 11 and 12.

8.1 A Simple Pure Exchange Overlapping Generations Model

Let's start by repeating the infinitely lived agent model to which we will compare the OLG model. Suppose there are I individuals that live forever. There is one nonstorable consumption good in each period. Individuals order consumption allocations according to

$$u_i(c_i) = \sum_{t=1}^{\infty} \beta_i^{t-1} U(c_t^i)$$

Note that agents start their lives at $t = 1$ to make this economy comparable to the OLG economies studied below. Agents have deterministic endowment streams $e^i = \{e_t^i\}_{t=0}^{\infty}$. Trade takes place at period 0. The standard definition of an Arrow-Debreu equilibrium goes like this:

Definition 71 *A competitive equilibrium are prices* $\{p_t\}_{t=0}^{\infty}$ *and allocations* $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i\in I}$ *such that*

1. *Given* $\{p_t\}_{t=0}^{\infty}$, *for all* $i \in I$, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ *solves* $\max_{c^i \geq 0} u_i(c_i)$ *subject to*

$$\sum_{t=0}^{\infty} p_t(c_t^i - e_t^i) \leq 0$$

2.

$$\sum_{i \in I} \hat{c}_t^i = \sum_{i \in I} e_t^i \text{ for all } t$$

What are the main shortcomings of this model that have lead to the development of the OLG model? The first criticism is that individuals apparently do not live forever, so that a model with finitely lived agents is needed. We will see later that we can give the infinitely lived agent model an interpretation in which individuals lived only for a finite number of periods, but, by having an altruistic bequest motive, act so as to maximize the utility of the entire dynasty, which in effect makes the planning horizon of the agent infinite. So infinite lives in itself are not as unsatisfactory as it may seem. But if people live forever, they don't undergo a life cycle with low-income youth, high income middle ages and retirement where labor income drops to zero. In the infinitely lived agent model every period is like the next (which makes it so useful since this stationarity renders dynamic programming techniques easily applicable). So in order to analyze issues like social security, the effect of taxes on retirement decisions, the distributive effects of taxes vs. government deficits, the effects of life-cycle saving on capital accumulation one needs a model in which agents experience a life cycle and in which people of different ages live at the same time in the economy. This is why the OLG model is a very useful tool for applied policy analysis. Because of its interesting (some say, pathological) theoretical properties, it is also an area of intense study among economic theorists.

8.1.1 Basic Setup of the Model

Let us describe the model formally now. Time is discrete, $t = 1, 2, 3, \ldots$ and the economy (but not its people) lives forever. In each period there is a single, nonstorable consumption good. In each time period a new generation (of measure 1) is born, which we index by its date of birth. People live for two periods and then die. By (e_t^t, e_{t+1}^t) we denote generation t's endowment of the consumption good in the first and second period of their live and by (c_t^t, c_{t+1}^t) we denote the consumption allocation of generation t. Hence in time t there are two generations alive, one old generation $t-1$ that has endowment e_t^{t-1} and consumption c_t^{t-1} and one young generation t that has endowment e_t^t and consumption c_t^t. In addition, in period 1 there is an initial old generation 0 that has endowment e_1^0 and consumes c_1^0. In some of our applications we will endow the initial generation with an amount of outside money[36] m. We will NOT assume

[36] Money that is, on net, an asset of the private economy, is "outside money". This includes fiat currency issued by the government. In contrast, inside money (such as bank deposits) is both an asset as well as a liability of the private sector (in the case of deposits an asset of the deposit holder, a liability to the bank).

$m \geq 0$. If $m \geq 0$, then m can be interpreted straightforwardly as fiat money, if $m < 0$ one should envision the initial old people having borrowed from some institution (which is, however, outside the model) and m is the amount to be repaid.

In the next Table 1 we demonstrate the demographic structure of the economy. Note that there are both an infinite number of periods as well as well as an infinite number of agents in this economy. This "double infinity" has been cited to be the major source of the theoretical peculiarities of the OLG model (prominently by Karl Shell).

Table 1

		Time				
G		1	2	…	t	$t+1$
e	0	(c_0^1, e_0^1)				
n	1	(c_1^1, e_1^1)	(c_2^1, e_2^1)			
e	$\vdots$			$\ddots$		
r	$t-1$				(c_t^{t-1}, e_t^{t-1})	
a	t				(c_t^t, e_t^t)	(c_{t+1}^t, e_{t+1}^t)
t.	$t-1$					$(c_{t+1}^{t+1}, e_{t+1}^{t+1})$

Preferences of individuals are assumed to be representable by an additively separable utility function of the form

$$u_t(c) = U(c_t^t) + \beta U(c_{t+1}^t)$$

and the preferences of the initial old generation is representable by

$$u_0(c) = U(c_1^0)$$

We shall assume that U is strictly increasing, strictly concave and twice continuously differentiable. This completes the description of the economy. Note that we can easily represent this economy in our formal Arrow-Debreu language from Chapter 7 since it is a standard pure exchange economy with infinite number of agents and the peculiar preference and endowment structure $e_s^t = 0$ for all $s \neq t, t+1$ and $u_t(c)$ only depending on c_t^t, c_{t+1}^t. You should complete the formal representation as a useful homework exercise.

The following definitions are straightforward

Definition 72 *An allocation is a sequence $c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}$. An allocation is feasible if $c_{t-1}^t, c_t^t \geq 0$ for all $t \geq 1$ and*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

An allocation $c_1^0, \{(c_t^t, c_{t+1}^t)\}_{t=1}^{\infty}$ is Pareto optimal if it is feasible and if there is no other feasible allocation $\hat{c}_0^1, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ such that

$$\begin{aligned} u_t(\hat{c}_t^t, \hat{c}_{t+1}^t) &\geq u_t(c_t^t, c_{t+1}^t) \text{ for all } t \geq 1 \\ u_0(\hat{c}_1^0) &\geq u_0(c_1^0) \end{aligned}$$

with strict inequality for at least one $t \geq 0$.

We now define an equilibrium for this economy in two different ways, depending on the market structure. Let p_t be the price of one unit of the consumption good at period t. In the presence of money (i.e. $m \neq 0$) we will take money to be the numeraire. This is important since we can only normalize the price of one commoditiy to 1, so with money no further normalizations are admissible. Of course, without money we are free to normalize the price of one other commodity. Keep this in mind for later. We now have the following

Definition 73 *Given m, an Arrow-Debreu equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^{\infty}$ and prices $\{p_t\}_{t=1}^{\infty}$ such that*

1. *Given $\{p_t\}_{t=1}^{\infty}$, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ solves*

$$\max_{(c_t^t, c_{t+1}^t) \geq 0} u_t(c_t^t, c_{t+1}^t) \tag{39}$$

$$s.t.\ p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} c_{t+1}^t \tag{40}$$

2. *Given p_1, $\hat{c}_1^0$ solves*

$$\max_{c_1^0} u_0(c_1^0)$$

$$s.t.\ p_1 c_1^0 \leq p_1 e_1^0 + m \tag{41}$$

3. *For all $t \geq 1$ (Resource Balance or goods market clearing)*

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1$$

As usual within the Arrow-Debreu framework, trading takes place in a hypothetical centralized market place at period 0 (even though the generations are not born yet).[37] There is an alternative definition of equilibrium that assumes sequential trading. Let r_{t+1} be the interest rate from period t to period $t+1$ and s_t^t be the savings of generation t from period t to period $t+1$. We will look at a slightly different form of assets in this section. Previously we dealt with one-period IOU's that had price q_t in period t and paid out one unit of the consumption good in $t+1$ (so-called zero bonds). Now we consider assets that cost one unit of consumption in period t and deliver $1 + r_{t+1}$ units tomorrow. Equilibria with these two different assets are obviously equivalent to each other, but the latter specification is easier to interpret if the asset at hand is fiat money.

We define a Sequential Markets (SM) equilibrium as follows:

Definition 74 *Given m, a sequential markets equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$ and interest rates $\{r_t\}_{t=1}^{\infty}$ such that*

[37] When naming this definition after Arrow-Debreu I make reference to the market *structure* that is envisioned under this definition of equilibrium. Others, including Geanakoplos, refer to a particular *model* when talking about Arrow-Debreu, the standard general equilibrium model encountered in micro with finite number of simultaneously living agents. I hope this does not cause any confusion.

1. *Given* $\{r_t\}_{t=1}^{\infty}$ *for each* $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ *solves*

$$\max_{(c_t^t, c_{t+1}^t) \geq 0, s_t^t} u_t(c_t^t, c_{t+1}^t)$$

$$s.t.\ c_t^t + s_t^t \leq e_t^t \tag{42}$$

$$c_{t+1}^t \leq e_{t+1}^t + (1 + r_{t+1}) s_t^t \tag{43}$$

2. *Given* r_1, $\hat{c}_1^0$ *solves*

$$\max_{c_1^0} u_0(c_1^0)$$

$$s.t.\ c_1^0 \leq e_1^0 + (1 + r_1)m$$

3. *For all* $t \geq 1$ *(Resource Balance or goods market clearing)*

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t \text{ for all } t \geq 1 \tag{44}$$

In this interpretation trade takes place sequentially in spot markets for consumption goods that open in each period. In addition there is an asset market through which individuals do their saving. Remember that when we wrote down the sequential formulation of equilibrium for an infinitely lived consumer model we had to add a shortsale constraint on borrowing (i.e. $s_t \geq -A$) in order to prevent Ponzi schemes, the continuous rolling over of higher and higher debt. This is not necessary in the OLG model as people live for a finite (two) number of periods (and we, as usual, assume perfect enforceability of contracts)

Given that the period utility function U is strictly increasing, the budget constraints (42) and (43) hold with equality. Take budget constraint (43) for generation t and (42) for generation $t+1$ and sum them up to obtain

$$c_{t+1}^t + c_{t+1}^{t+1} + s_{t+1}^{t+1} = e_{t+1}^t + e_{t+1}^{t+1} + (1 + r_{t+1}) s_t^t$$

Now use equation (44) to obtain

$$s_{t+1}^{t+1} = (1 + r_{t+1}) s_t^t$$

Doing the same manipulations for generation 0 and 1 gives

$$s_1^1 = (1 + r_1)m$$

and hence, using repeated substitution one obtains

$$s_t^t = \Pi_{\tau=1}^t (1 + r_\tau)m \tag{45}$$

This is the market clearing condition for the asset market: the amount of saving (in terms of the period t consumption good) has to equal the value of the outside supply of assets, $\Pi_{\tau=1}^t (1 + r_\tau)m$. Strictly speaking one should include condition (45) in the definition of equilibrium. By Walras' law however, either the asset market or the good market equilibrium condition is redundant.

There is an obvious sense in which equilibria for the Arrow-Debreu economy (with trading at period 0) are equivalent to equilibria for the sequential markets economy. For $r_{t+1} > -1$ combine (42) and (43) into

$$c_t^t + \frac{c_{t+1}^t}{1+r_{t+1}} = e_t^t + \frac{e_{t+1}^t}{1+r_{t+1}}$$

Divide (40) by $p_t > 0$ to obtain

$$c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t = e_t^t + \frac{p_{t+1}}{p_t} e_{t+1}^t$$

Furthermore divide (41) by $p_1 > 0$ to obtain

$$c_1^0 \leq e_1^0 + \frac{m}{p_1}$$

We then can straightforwardly prove the following proposition

Proposition 75 *Let allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t)\}_{t=1}^\infty$ and prices $\{p_t\}_{t=1}^\infty$ constitute an Arrow-Debreu equilibrium with $p_t > 0$ for all $t \geq 1$. Then there exists a corresponding sequential market equilibrium with allocations $\tilde{c}_1^0, \{(\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{s}_t^t)\}_{t=1}^\infty$ and interest rates $\{r_t\}_{t=1}^\infty$ with*

$$\begin{aligned} \tilde{c}_t^{t-1} &= \hat{c}_t^{t-1} \text{ for all } t \geq 1 \\ \tilde{c}_t^t &= \hat{c}_t^t \text{ for all } t \geq 1 \end{aligned}$$

Furthermore, let allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^\infty$ and interest rates $\{r_t\}_{t=1}^\infty$ constitute a sequential market equilibrium with $r_t > -1$ for all $t \geq 0$. Then there exists a corresponding Arrow-Debreu equilibrium with allocations $\tilde{c}_1^0, \{(\tilde{c}_t^t, \tilde{c}_{t+1}^t)\}_{t=1}^\infty$ and prices $\{p_t\}_{t=1}^\infty$ such that

$$\begin{aligned} \tilde{c}_t^{t-1} &= \hat{c}_t^{t-1} \text{ for all } t \geq 1 \\ \tilde{c}_t^t &= \hat{c}_t^t \text{ for all } t \geq 1 \end{aligned}$$

Proof. The proof is similar to the infinite horizon counterpart. Given equilibrium Arrow-Debreu prices $\{p_t\}_{t=1}^\infty$ define interest rates as

$$\begin{aligned} 1 + r_{t+1} &= \frac{p_t}{p_{t+1}} \\ 1 + r_1 &= \frac{1}{p_1} \end{aligned}$$

and savings

$$\tilde{s}_t^t = e_t^t - \hat{c}_t^t$$

It is straightforward to verify that the allocations and prices so constructed constitute a sequential markets equilibrium.

Given equilibrium sequential markets interest rates $\{r_t\}_{t=1}^{\infty}$ define Arrow-Debreu prices by

$$\begin{aligned} p_1 &= \frac{1}{1+r_1} \\ p_{t+1} &= \frac{p_t}{1+r_{t+1}} \end{aligned}$$

Again it is straightforward to verify that the prices and allocations so constructed form an Arrow-Debreu equilibrium. ■

Note that the requirement on interest rates is weaker for the OLG version of this proposition than for the infinite horizon counterpart. This is due to the particular specification of the no-Ponzi condition used. A less stringent condition still ruling out Ponzi schemes would lead to a weaker condtion in the proposition for the infinite horizon economy also.

Also note that with this equivalence we have that

$$\Pi_{\tau=1}^{t}(1+r_\tau)m = \frac{m}{p_t}$$

so that the asset market clearing condition for the sequential markets economy can be written as

$$p_t s_t^t = m$$

i.e. the demand for assets (saving) equals the outside supply of assets, m. Note that the demanders of the assets are the currently young whereas the suppliers are the currently old people. From the equivalence we can also see that the return on the asset (to be interpreted as money) equals

$$\begin{aligned} 1+r_{t+1} &= \frac{p_t}{p_{t+1}} = \frac{1}{1+\pi_{t+1}} \\ (1+r_{t+1})(1+\pi_{t+1}) &= 1 \\ r_{t+1} &\approx -\pi_{t+1} \end{aligned}$$

where π_{t+1} is the inflation rate from period t to $t+1$. As it should be, the real return on money equals the negative of the inflation rate.

8.1.2 Analysis of the Model Using Offer Curves

Unless otherwise noted in this subsection we will focus on Arrow-Debreu equilibria. Gale (1973) developed a nice way of analyzing the equilibria of a two-period OLG economy graphically, using offer curves. First let us assume that the economy is stationary in that $e_t^t = w_1$ and $e_{t+1}^t = w_2$, i.e. the endowments are time invariant. For given $p_t, p_{t+1} > 0$ let by $c_t^t(p_t, p_{t+1})$ and $c_{t+1}^t(p_t, p_{t+1})$ denote the solution to maximizing (39) subject to (40) for all $t \geq 1$. Given or assumptions this solution is unique. Let the excess demand functions y and z be defined by

$$\begin{aligned} y(p_t, p_{t+1}) &= c_t^t(p_t, p_{t+1}) - e_t^t \\ &= c_t^t(p_t, p_{t+1}) - w_1 \\ z(p_t, p_{t+1}) &= c_{t+1}^t(p_t, p_{t+1}) - w_2 \end{aligned}$$

These two functions summarize, for given prices, all implications that consumer optimization has for equilibrium allocations. Note that from the Arrow-Debreu budget constraint it is obvious that y and z only depend on the ratio $\frac{p_{t+1}}{p_t}$, but not on p_t and p_{t+1} separately (this is nothing else than saying that the excess demand functions are homogeneous of degree zero in prices, as they should be). Varying $\frac{p_{t+1}}{p_t}$ between 0 and ∞ (not inclusive) one obtains a locus of optimal excess demands in (y, z) space, the so called offer curve. Let us denote this curve as

$$(y, f(y)) \tag{46}$$

where it is understood that f can be a correspondence, i.e. multi-valued. A point on the offer curve is an optimal excess demand function for *some* $\frac{p_{t+1}}{p_t} \in (0, \infty)$. Also note that since $c_t^t(p_t, p_{t+1}) \geq 0$ and $c_{t+1}^t(p_t, p_{t+1}) \geq 0$ the offer curve obviously satisfies $y(p_t, p_{t+1}) \geq -w_1$ and $z(p_t, p_{t+1}) \geq -w_2$. Furthermore, since the optimal choices obviously satisfy the budget constraint, i.e.

$$\begin{aligned} p_t y(p_t, p_{t+1}) + p_{t+1} z(p_t, p_{t+1}) &= 0 \\ \frac{z(p_t, p_{t+1})}{y(p_t, p_{t+1})} &= -\frac{p_t}{p_{t+1}} \end{aligned} \tag{47}$$

Equation (47) is an equation in the two unknowns (p_t, p_{t+1}) for a given $t \geq 1$. Obviously $(y, z) = (0, 0)$ is on the offer curve, as for appropriate prices (which we will determine later) no trade is the optimal trading strategy. Equation (47) is very useful in that for a given point on the offer curve $(y(p_t, p_{t+1}), z(p_t, p_{t+1}))$ in y-z space with $y(p_t, p_{t+1}) \neq 0$ we can immediately read off the price ratio at which these are the optimal demands. Draw a straight line through the point (y, z) and the origin; the slope of that line equals $-\frac{p_t}{p_{t+1}}$. One should also note that if $y(p_t, p_{t+1})$ is negative, then $z(p_t, p_{t+1})$ is positive and vice versa. Let's look at an example

Example 76 *Let $w_1 = \varepsilon$, $w_2 = 1 - \varepsilon$, with $\varepsilon > 0$. Also let $U(c) = \ln(c)$ and $\beta = 1$. Then the first order conditions imply*

$$p_t c_t^t = p_{t+1} c_{t+1}^t \tag{48}$$

and the optimal consumption choices are

$$c_t^t(p_t, p_{t+1}) = \frac{1}{2}\left(\varepsilon + \frac{p_{t+1}}{p_t}(1 - \varepsilon)\right) \tag{49}$$

$$c_{t+1}^t(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_t}{p_{t+1}}\varepsilon + (1 - \varepsilon)\right) \tag{50}$$

the excess demands are given by

$$y(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_{t+1}}{p_t}(1 - \varepsilon) - \varepsilon\right) \tag{51}$$

$$z(p_t, p_{t+1}) = \frac{1}{2}\left(\frac{p_t}{p_{t+1}}\varepsilon - (1 - \varepsilon)\right) \tag{52}$$

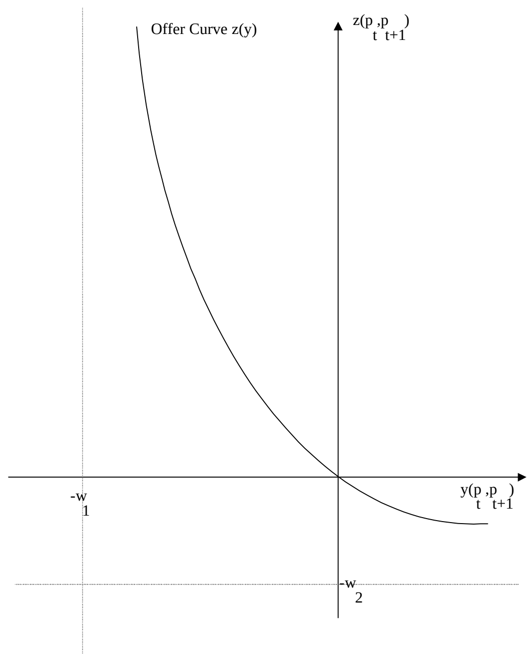


Figure 8:

Note that as $\frac{p_{t+1}}{p_t} \in (0, \infty)$ *varies, y varies between* $-\frac{\varepsilon}{2}$ *and* ∞ *and z varies between* $-\frac{(1-\varepsilon)}{2}$ *and* ∞*. Solving z as a function of y by eliminating* $\frac{p_{t+1}}{p_t}$ *yields*

$$z = \frac{\varepsilon(1-\varepsilon)}{4y+2\varepsilon} - \frac{1-\varepsilon}{2} \text{ for } y \in (-\frac{\varepsilon}{2}, \infty) \tag{53}$$

This is the offer curve $(y, z) = (y, f(y))$*. We draw the offer curve in Figure 8*

The discussion of the offer curve takes care of the first part of the equilibrium definition, namely optimality. It is straightforward to express goods market clearing in terms of excess demand functions as

$$y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0 \tag{54}$$

Also note that for the initial old generation the excess demand function is given by

$$z_0(p_1, m) = \frac{m}{p_1}$$

so that the goods market equilibrium condition for the first period reads as

$$y(p_1, p_2) + z_0(p_1, m) = 0 \tag{55}$$

Graphically in (y, z)-space equations (54) and (55) are straight lines through the origin with slope -1. All points on this line are resource feasible. We therefore have the following procedure to find equilibria for this economy for a given initial endowment of money m of the initial old generation, using the offer curve (46) and the resource feasibility constraints (54) and (55).

1. Pick an initial price p_1 (note that this is NOT a normalization as in the infinitely lived agent model since the value of p_1 determines the real value of money $\frac{m}{p_1}$ the initial old generation is endowed with; we have already normailzed the price of money). Hence we know $z_0(p_1, m)$. From (55) this determines $y(p_1, p_2)$.

2. From the offer curve (46) we determine $z(p_1, p_2) \in f(y(p_1, p_2))$. Note that if f is a correspondence then there are multiple choices for z.

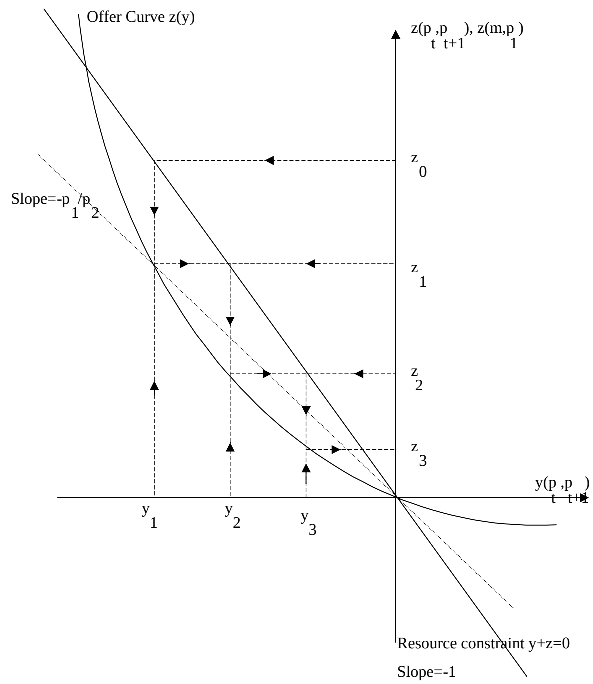
3. Once we know $z(p_1, p_2)$, from (54) we can find $y(p_2, p_3)$ and so forth. In this way we determine the entire equilibrium consumption allocation

$$\begin{aligned} c_1^0 &= z_0(p_1, m) + w_2 \\ c_t^t &= y(p_t, p_{t+1}) + w_1 \\ c_{t+1}^t &= z(p_t, p_{t+1}) + w_2 \end{aligned}$$

4. Equilibrium prices can then be found, given p_1 from equation (47). Any initial p_1 that induces, in such a way, sequences $c_1^0, \{(c_t^t, c_{t+1}^t), p_t\}_{t=1}^{\infty}$ such that the consumption sequence satisfies $c_t^{t-1}, c_t^t \geq 0$ is an equilibrium for given money stock. This already indicates the possibility of a lot of equilibria for this model, a fact that we will demonstrate below.

This algorithm can be demonstrated graphically using the offer curve diagram. We add the line representing goods market clearing, equation (54). In the (y, z)-plane this is a straight line through the origin with slope -1. This line intersects the offer curve at least once, namely at the origin. Unless we have the degenerate situation that the offer curve has slope -1 at the origin, there is (at least) one other intersection of the offer curve with the goods clearing line. These intersection will have special significance as they will represent stationary equilibria. As we will see, there is a load of other equilibria as well. We will first describe the graphical procedure in general and then look at some examples. See Figure 9.

Given any m (for concreteness let $m > 0$) pick $p_1 > 0$. This determines $z_0 = \frac{m}{p_1} > 0$. Find this quantity on the z-axis, representing the excess demand of the initial old generation. From this point on the z-axis go horizontally to the goods market line, from there down to the y-axis. The point on the y-axis represents the excess demand function of generation 1 when young. From this



Offer Curve z(y)
z(p_t, p_{t+1}), z(m,p_1)
z_0
Slope=-p_1/p_2
z_1
z_2
z_3
y(p_t, p_{t+1})
y_1
y_2
y_3
Resource constraint y+z=0
Slope=-1

Figure 9:

point $y_1 = y(p_1, p_2)$ go vertically to the offer curve, then horizontally to the z-axis. The resulting point $z_1 = z(p_1, p_2)$ is the excess demand of generation 1 when old. Then back horizontally to the goods market clearing condition and down yields $y_2 = y(p_2, p_3)$, the excess demand for the second generation and so on. This way the entire equilibrium consumption allocation can be constructed. Equilibrium prices are easily found from equilibrium allocations with (47), given p_1. In such a way we construct an entire equilibrium graphically.

Let's now look at some example.

Example 77 *Reconsider the example with isoelastic utility above. We found the offer curve to be*

$$z = \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2} \text{ for } y \in (-\frac{\varepsilon}{2}, \infty)$$

The goods market equilibrium condition is

$$y + z = 0$$

Now let's construct an equilibrium for the case $m = 0$, for zero supply of outside money. Following the procedure outlined above we first find the excess demand function for the initial old generation $z_0(m, p_1) = 0$ for all $p_1 > 0$. Then from goods market $y(p_1, p_2) = -z_0(m, p_1) = 0$. From the offer curve

$$\begin{aligned} z(p_1, p_2) &= \frac{\varepsilon(1-\varepsilon)}{4y(p_1, p_2) + 2\varepsilon} - \frac{1-\varepsilon}{2} \\ &= \frac{\varepsilon(1-\varepsilon)}{2\varepsilon} - \frac{1-\varepsilon}{2} \\ &= 0 \end{aligned}$$

and continuing we find $z(p_t, p_{t+1}) = y(p_t, p_{t+1}) = 0$ for all $t \geq 1$. This implies that the equilibrium allocation is $c_t^{t-1} = 1 - \varepsilon, c_t^t = \varepsilon$. In this equilibrium every consumer eats his endowment in each period and no trade between generations takes place. We call this equilibrium the autarkic equilibrium. Obviously we can't determine equilibrium prices from equation (47). However, the first order conditions imply that

$$\frac{p_{t+1}}{p_t} = \frac{c_t^t}{c_{t+1}^t} = \frac{\varepsilon}{1-\varepsilon}$$

For $m = 0$ we can, without loss of generality, normalize the price of the first period consumption good $p_1 = 1$. Note again that only for $m = 0$ this normalization is innocuous, since it does not change the real value of the stock of outside money that the initial old generation is endowed with. With this normalization the sequence $\{p_t\}_{t=1}^{\infty}$ defined as

$$p_t = \left(\frac{\varepsilon}{1-\varepsilon}\right)^{t-1}$$

together with the autarkic allocation form an (Arrow-Debreu)-equilibrium. Obviously any other price sequence $\{\bar{p}_t\}$ with $\bar{p}_t = \alpha p_t$ for any $\alpha > 1$, is also an equilibrium price sequence supporting the autarkic allocation as equilibrium. This is not, however, what we mean by the possibility of a continuum of equilibria in OLG-model, but rather the usual feature of standard competitive equilibria that the equilibrium prices are only determined up to one normalization. In fact, for this example with $m = 0$, the autarkic equilibrium is the unique equilibrium for this economy.[38] *This is easily seen. Since the initial old generation has no money, only its endowments $1-\varepsilon$, there is no way for them to consume more than their endowments. Obviously they can always assure to consume at least their endowments by not trading, and that is what they do for any $p_1 > 0$ (obviously $p_1 \leq 0$ is not possible in equilibrium). But then from the resource constraint it follows that the first young generation must consume their endowments when young. Since they haven't saved anything, the best they can do when old is to consume their endowment again. But then the next young generation is forced to consume their endowments and so forth. Trade breaks down completely. For this allocation to be an equilibrium prices must be such that at these prices all generations actually find it optimal not to trade, which yields the prices below.*[39]

Note that in the picture the second intersection of the offer curve with the resource constraint (the first is at the origin) occurs in the forth orthant. This need not be the case. If the slope of the offer curve at the origin is less than one, we obtain the picture above, if the slope is bigger than one, then the second intersection occurs in the second orthant. Let us distinguish between these two cases more carefully. In general, the price ratio supporting the autarkic equilibrium satisfies

$$\frac{p_t}{p_{t+1}} = \frac{U'(e_t^t)}{\beta U'(e_{t+1}^t)} = \frac{U'(w_1)}{\beta U'(w_2)}$$

and this ratio represents the slope of the offer curve at the origin. With this in mind define the autarkic interest rate (remember our equivalence result from

[38] The fact that the autarkic is the only equilibrium is specific to pure exchange OLG-models with agents living for only two periods. Therefore Samuelson (1958) considered three-period lived agents for most of his analysis.

[39] If you look at Sargent and Ljungquist (1999), Chapter 8, you will see that they claim to construct several equilibria for exactly this example. Note, however, that their equilibrium definition has as feasibility constraint

$$c_t^{t-1} + c_t^t \leq e_t^{t-1} + e_t^t$$

and all the equilibria apart from the autarkic one constructed above have the feature that for $t = 1$

$$c_1^0 + c_1^1 < e_1^0 + e_1^1$$

which violate feasibility in the way we have defined it. Personally I find the free disposal assumption not satisfactory; it makes, however, their life easier in some of the examples to follow, whereas in my discussion I need more handwaving. You'll see.

above) as

$$1 + \bar{r} = \frac{U'(w_1)}{\beta u'(w_2)}$$

Gale (1973) has invented the following terminology: when $\bar{r} < 0$ he calls this the Samueson case, whereas when $\bar{r} \geq 0$ he calls this the classical case.[40] As it will turn out and will be demonstrated below autarkic equilibria are not Pareto optimal in the Samuelson case whereas they are in the classical case.

8.1.3 Inefficient Equilibria

The preceding example can also serve to demonstrate our first major feature of OLG economies that sets it apart from the standard infinitely lived consumer model with finite number of agents: competitive equilibria may be not be Pareto optimal. For economies like the one defined at the beginning of the section the two welfare theorems were proved and hence equilibria are Pareto optimal. Now let's see that the equilibrium constructed above for the OLG model may not be.

Note that in the economy above the aggregate endowment equals to 1 in each period. Also note that then the value of the aggregate endowment at the equilibrium prices, given by $\sum_{t=1}^{\infty} p_t$. Obviously, if $\varepsilon < 0.5$, then this sum converges and the value of the aggregate endowment is finite, whereas if $\varepsilon \geq 0.5$, then the value of the aggregate endowment is infinite. Whether the value of the aggregate endowment is infinite has profound implications for the welfare properties of the competitive equilibrium. In particular, using a similar argument as in the standard proof of the first welfare theorem you can show (and will do so in the homework) that if $\sum_{t=1}^{\infty} p_t < \infty$, then the competitive equilibrium allocation for this economy (and in general for any pure exchange OLG economy) is Pareto-efficient. If, however, the value of the aggregate endowment is infinite (at the equilibrium prices), then the competitive equilibrium MAY not be Pareto optimal. In our current example it turns out that if $\varepsilon > 0.5$, then the autarkic equilibrium is not Pareto efficient, whereas if $\varepsilon = 0.5$ it is. Since interest rates are defined as

$$r_{t+1} = \frac{p_t}{p_{t+1}} - 1$$

$\varepsilon\dot{<}0.5$ implies $r_{t+1} = \frac{1-\varepsilon}{\varepsilon} - 1 = \frac{1}{\varepsilon} - 2$. Hence $\varepsilon < 0.5$ implies $r_{t+1} > 0$ (the classical case) and $\varepsilon \geq 0.5$ implies $r_{t+1} < 0$. (the Samuelson case). Inefficiency

[40] More generally, the Samuelson case is defined by the condition that savings of the young generation be positive at an interest rate equal to the population growth rate n. So far we have assumed $n = 0$, so the Samuelson case requires saving to be positive at zero interest rate. We stated the condition as $\bar{r} < 0$. But if the interest rate at which the young don't save (the autarkic allocation) is smaller than zero, then at the higher interest rate of zero they will save a positive amount, so that we can define the Samuelson case as in the text, provided that savings are strictly increasing in the interest rate. This in turn requires the assumption that first and second period consumption are strict gross substitutes, so that the offer curve is not backward-bending. In the homework you will encounter an example in which this assumption is not satisfied.

is therefore associated with low (negative interest rates). In fact, Balasko and Shell (1980) show that the autarkic equilibrium is Pareto optimal if and only if

$$\sum_{t=1}^{\infty}\prod_{\tau=1}^{t}(1+r_{\tau+1})=+\infty$$

where $\{r_{t+1}\}$ is the sequence of autarkic equilibrium interest rates.[41] Obviously the above equation is satisfied if and only if $\varepsilon \leq 0.5$.

Let us briefly demonstrate the first claim (a more careful discussion is left for the homework). To show that for $\varepsilon > 0.5$ the autarkic allocation (which is the unique equilibrium allocation) is not Pareto optimal it is sufficient to find another feasible allocation that Pareto-dominates it. Let's do this graphically in Figure 10. The autarkic allocation is represented by the origin (excess demand functions equal zero). Consider an alternative allocation represented by the intersection of the offer curve and the resource constraint. We want to argue that this point Pareto dominates the autarkic allocation. First consider an arbitrary generation $t \geq 1$. Note that the indifference curve through any point must lie (locally) to the inside of the offer curve. From (47) we saw that the price ratio $\frac{p_t}{p_{t+1}}$ at which a point on the offer curve is the optimal choice is a line through the origin and through the point of question. This line represents nothing else but the budget line at the price ratio $\frac{p_t}{p_{t+1}}$. Since the point on the offer curve is utility maximizing choices given the prices the indifference curve through the point must lie tangent above the line through the point and the origin. Any other point on this line (including the origin) must be weakly worse

[41] Rather than a formal proof (which is quite involved), let's develop some intuition for why low interest rates are associated with inefficiency. Take the autarkic allocation and try to construct a Pareto improvement. In particular, give additional $\delta_0 > 0$ units of consumption to the initial old generation. This obviously improves this generation's life. From resource feasibilty this requires taking away δ_0 from generation 1 in their first period of life. To make them not worse of they have to recieve δ_1 in additional consumption in their second period of life, with δ_1 satisfying

$$\delta_0 U'(e_1^1)=\delta_1\beta U'(e_2^1)$$

or

$$\begin{aligned}\delta_1 &= \delta_0\frac{\beta U'(e_2^1)}{U'(e_1^1)}\\ &= \delta_0(1+r_2)>0\end{aligned}$$

and in general

$$\delta_t=\delta_0\prod_{\tau=1}^{t}(1+r_{\tau+1})$$

are the required transfers in the second period of generation t's life to compensate for the reduction of first period consumption. Obviously such a scheme does not work if the economy ends at fine time T since the last generation (that lives only through youth) is worse off. But as our economy extends forever, such an intergenerational transfer scheme is feasible provided that the δ_t don't grow too fast, i.e. if interest rates are sufficiently small. But if such a transfer scheme is feasible, then we found a Pareto improvement over the original autarkic allocation, and hence the autarkic equilibrium allocation is not Pareto efficient.

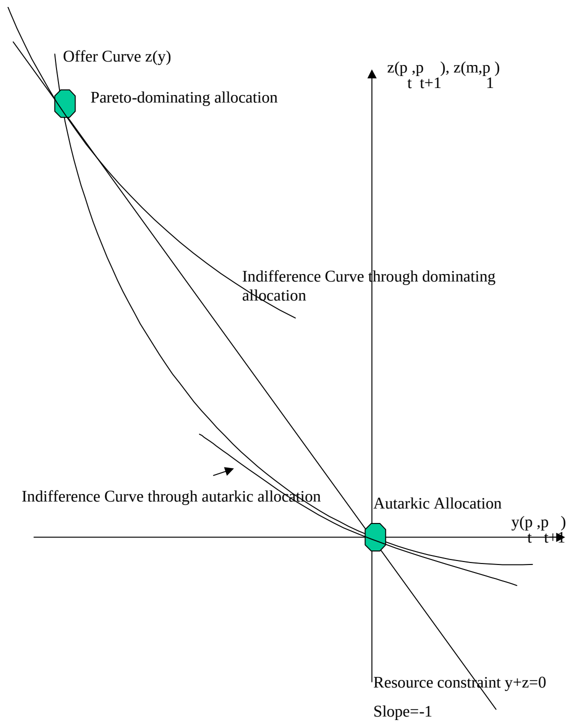


Figure 10:

than this point at given prices $\frac{p_t}{p_{t+1}}$. If we take $p_t = p_{t+1}$ this demonstrates that the alternative point (which is both on the offer curve as well as the resource constraint, the line with slope -1) is at least as good as the autarkic allocation for all generations $t \geq 1$. What about the initial old generation? In the autarkic allocation it has $c_1^0 = 1 - \varepsilon$, or $z_0 = 0$. In the new allocation it has $z_0 > 0$ as shown in the figure, so the initial old generation is strictly better off in this new allocation. Hence the alternative allocation Pareto-dominates the autarkic equilibrium allocation, which shows that this allocation is not Pareto-optimal. In the homework you are asked to make this argument rigorous by actually computing the alternative allocation and then arguing that it Pareto-dominates the autarkic equilibrium.

What in our graphical argument hinges on the assumption that $\varepsilon > 0.5$. Remember that for $\varepsilon \leq 0.5$ we have said that the autarkic allocation is actually Pareto optimal. It turns out that for $\varepsilon < 0.5$, the intersection of the resource constraint and the offer curve lies in the fourth orthant instead of in the second as in Figure 10. It is still the case that every generation $t \geq 1$ at least weakly prefers the alternative to the autarkic allocation. Now, however, this alternative allocation has $z_0 < 0$, which makes the initial old generation worse off than in the autarkic allocation, so that the argument does not work. Finally, for $\varepsilon = 0.5$ we have the degenerate situation that the slope of the offer curve at the origin is -1, so that the offer curve is tangent to the resource line and there is no second intersection. Again the argument does not work and we can't argue that the autarkic allocation is not Pareto optimal. It is an interesting optional exercise to show that for $\varepsilon = 0.5$ the autarkic allocation is Pareto optimal.

Now we want to demonstrate the second and third feature of OLG models that set it apart from standard Arrow-Debreu economies, namely the possibility of a continuum of equilibria and the fact that outside money may have positive value. We will see that, given the way we have defined our equilibria, these two issues are intimately linked. So now let us suppose that $m \neq 0$. In our discussion we will assume that $m > 0$, the situation for $m < 0$ is symmetric. We first want to argue that for $m > 0$ the economy has a continuum of equilibria, not of the trivial sort that only prices differ by a constant, but that allocations differ across equilibria. Let us first look at equilibria that are stationary in the following sense:

Definition 78 *An equilibrium is stationary if* $c_t^{t-1} = c^o$, $c_t^t = c^y$ *and* $\frac{p_{t+1}}{p_t} = a$, *where a is a constant.*

Given that we made the assumption that each generation has the same endowment structure a stationary equilibrium necessarily has to satisfy $y(p_t, p_{t+1}) = y$, $z_0(m, p_1) = z(p_t, p_{t+1}) = z$ for all $t \geq 1$. From our offer curve diagram the only candidates are the autarkic equilibrium (the origin) and any other allocations represented by intersections of the offer curve and the resource line. We will discuss the possibility of an autarkic equilibrium with money later. With respect to other stationary equilibria, they all have to have prices $\frac{p_{t+1}}{p_t} = 1$, with p_1 such that $(\frac{m}{p_1}, -\frac{m}{p_1})$ is on the offer curve. For our previous example, for any $m \neq 0$ we find the stationary equilibrium by solving for the intersection of offer curve and resource line

$$\begin{aligned} y + z &= 0 \\ z &= \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2} \end{aligned}$$

This yields a second order polynomial in y

$$-y = \frac{\varepsilon(1-\varepsilon)}{4y + 2\varepsilon} - \frac{1-\varepsilon}{2}$$

whose one solution is $y = 0$ (the autarkic allocation) and the other solution is $y = \frac{1}{2} - \varepsilon$, so that $z = -\frac{1}{2} + \varepsilon$. Hence the corresponding consumption allocation has

$$c_t^{t-1} = c_t^t = \frac{1}{2} \text{ for all } t \geq 1$$

In order for this to be an equilibrium we need

$$\frac{1}{2} = c_1^0 = (1-\varepsilon) + \frac{m}{p_1}$$

hence $p_1 = \frac{m}{\varepsilon - 0.5} > 0$. Therefore a stationary equilibrium (apart from autarky) only exists for $m > 0$ and $\varepsilon > 0.5$ or $m < 0$ and $\varepsilon < 0.5$. Also note that the choice of p_1 is not a matter of normalization: any multiple of p_1 will not yield a stationary equilibrium. The equilibrium prices supporting the stationary allocation have $p_t = p_1$ for all $t \geq 1$. Finally note that this equilibrium, since it features $\frac{p_{t+1}}{p_t} = 1$, has an inflation rate of $\pi_{t+1} = -r_{t+1} = 0$. It is exactly this equilibrium allocation that we used to prove that, for $\varepsilon > 0.5$, the autarkic equilibrium is not Pareto-efficient.

How about the autarkic allocation? Obviously it is stationary as $c_t^{t-1} = 1-\varepsilon$ and $c_t^t = \varepsilon$ for all $t \geq 1$. But can it be made into an equilibrium if $m \neq 0$. If we look at the sequential markets equilibrium definition there is no problem: the budget constraint of the initial old generation reads

$$c_1^0 = 1 - \varepsilon + (1 + r_1)m$$

So we need $r_1 = -1$. For all other generations the same arguments as without money apply and the interest sequence satisfying $r_1 = -1$, $r_{t+1} = \frac{1-\varepsilon}{\varepsilon} - 1$ for all $t \geq 1$, together with the autarkic allocation forms a sequential market equilibrium. In this equilibrium the stock of outside money, m, is not valued: the initial old don't get any goods in exchange for it and future generations are not willing to ever exchange goods for money, which results in the autarkic, no-trade situation. To make autarky an Arrow-Debreu equilibrium is a bit more problematic. Again from the budget constraint of the initial old we find

$$c_1^0 = 1 - \varepsilon + \frac{m}{p_1}$$

which, for autarky to be an equilibrium requires $p_1 = \infty$, i.e. the price level is so high in the first period that the stock of money de facto has no value. Since for all other periods we need $\frac{p_{t+1}}{p_t} = \frac{\varepsilon}{1-\varepsilon}$ to support the autarkic allocation, we have the obscure requirement that we need price *levels* to be infinite with well-defined finite price *ratios*. This is unsatisfactory, but there is no way around it unless we a) change the equilibrium definition (see Sargent and Ljungquist) or b) let the economy extend from the infinite past to the infinite future (instead of starting with an initial old generation, see Geanakoplos) or c) treat money somewhat as a residual, as something almost endogenous (see Kehoe) or d) make some consumption good rather than money the numeraire (with nonmonetary equilibria corresponding to situations in which money has a price of zero in terms of real consumption goods). For now we will accept autarky as an equilibrium even with money and we will treat it as identical to the autarkic equilibrium without money (because indeed in the sequential markets formulation only r_1 changes and in the Arrow Debreu formulation only p_1 changes, although in an unsatisfactory fashion).

8.1.4 Positive Valuation of Outside Money

In our construction of the nonautarkic stationary equilibrium we have already demonstrated our second main result of OLG models: outside money may have positive value. In that equilibrium the initial old had endowment $1-\varepsilon$ but consumed $c_1^0 = \frac{1}{2}$. If $\varepsilon > \frac{1}{2}$, then the stock of outside money, m, is valued in equilibrium in that the old guys can exchange m pieces of intrinsically worthless paper for $\frac{m}{p_1} > 0$ units of period 1 consumption goods.[42] The currently young generation accepts to transfer some of their endowment to the old people for pieces of paper because they expect (correctly so, in equilibrium) to exchange these pieces of paper against consumption goods when they are old, and hence to achieve an intertemporal allocation of consumption goods that dominates the autarkic allocation. Without the outside asset, again, this economy can do nothing else but remain in the possibly dismal state of autarky (imagine $\varepsilon = 1$ and log-utility). This is why the social contrivance of money is so useful in this economy. As we will see later, other institutions (for example a pay-as-you-go social security system) may achieve the same as money.

Before we demonstrate that, apart from stationary equilibria (two in the example, usually at least only a finite number) there may be a continuum of other, nonstationary equilibria we take a little digression to show for the general infinitely lived agent endowment economies set out at the beginning of this section money cannot have positive value in equilibrium.

Proposition 79 *In pure exchange economies with a finite number of infinitely lived agents there cannot be an equilibrium in which outside money is valued.*

Proof. Suppose, to the contrary, that there is an equilibrium $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}, \{\hat{p}_t\}_{t=1}^{\infty}$ for initial endowments of outside money $(m^i)_{i\in I}$ such that $\sum_{i\in I} m^i \neq 0$. Given the assumption of local nonsatiation each consumer in equilibrium satisfies the Arrow-Debreu budget constraint with equality

$$\sum_{t=1}^{\infty} \hat{p}_t \hat{c}_t^i = \sum_{t=1} \hat{p}_t e_t^i + m^i < \infty$$

Summing over all individuals $i \in I$ yields

$$\sum_{t=1}^{\infty} \hat{p}_t \sum_{i\in I} \left(\hat{c}_t^i - e_t^i\right) = \sum_{i\in I} m^i$$

But resource feasibility requires $\sum_{i\in I} \left(\hat{c}_t^i - e_t^i\right) = 0$ for all $t \geq 1$ and hence

$$\sum_{i\in I} m^i = 0$$

[42] In finance lingo money in this equilibrium is a "bubble". The fundamental value of an assets is the value of its dividends, evaluated at the equilibrium Arrow-Debreu prices. An asset is (or has) a bubble if its price does not equal its fundamental value. Obviuosly, since money doesn't pay dividends, its fundamental value is zero and the fact that it is valued positively in equilibrium makes it a bubble.

a contradiction. This shows that there cannot exist an equilibrium in this type of economy in which outside money is valued in equilibrium. Note that this result applies to a much wider class of standard Arrow-Debreu economies than just the pure exchange economies considered in this section. ■

Hence we have established the second major difference between the standard Arrow-Debreu general equilibrium model and the OLG model.

8.1.5 Continuum of Equilibria

We will now go ahead and demonstrate the third major difference, the possibility of a whole continuum of equilibria in OLG models. We will restrict ourselves to the specific example. Again suppose $m > 0$ and $\varepsilon > 0.5$.[43] For any p_1 such that $\frac{m}{p_1} < \varepsilon - \frac{1}{2} > 0$ we can construct an equilibrium using our geometric method before. From the picture it is clear that all these equilibria have the feature that the equilibrium allocations over time converge to the autarkic allocation, with $z_0 > z_1 > z_2 > \ldots z_t > 0$ and $\lim_{t\to\infty} z_t = 0$ and $0 > y_t > \ldots y_2 > y_1$ with $\lim_{t\to\infty} y_t = 0$. We also see from the figure that, since the offer curve lies below the -45^0-line for the part we are concerned with that $\frac{p_1}{p_2} < 1$ and $\frac{p_t}{p_{t+1}} < \frac{p_{t-1}}{p_t} < \ldots < \frac{p_1}{p_2} < 1$, implying that prices are increasing with $\lim_{t\to\infty} p_t = \infty$. Hence all the nonstationary equilibria feature inflation, although the inflation rate is bounded above by $\pi_\infty = -r_\infty = 1 - \frac{1-\varepsilon}{\varepsilon} = 2 - \frac{1}{\varepsilon} > 0$. The real value of money, however, declines to zero in the limit.[44] Note that, although all nonstationary equilibria so constructed in the limit converge to the same allocation (autarky), they differ in the sense that at any finite t, the consumption allocations and price ratios (and levels) differ across equilibria. Hence there is an entire continuum of equilibria, indexed by $p_1 \in (\frac{m}{\varepsilon - 0.5}, \infty)$. These equilibria are arbitrarily close to each other. This is again in stark contrast to standard Arrow-Debreu economies where, generically, the set of equilibria is finite and all equilibria are locally unique.[45] For details consult Debreu (1970) and the references therein.

Note that, if we are in the *Samuelson case* $\bar{r} < 0$, then (and only then) all these equilibria are Pareto-ranked.[46] Let the equilibria be indexed by p_1. One can show, by similar arguments that demonstrated that the autarkic equilibrium is not Pareto optimal, that these equilibria are Pareto-ranked: let $p_1, \hat{p}_1 \in (\frac{m}{\varepsilon - 0.5}, \infty)$ with $p_1 > \hat{p}_1$, then the equilibrium corresponding to $\hat{p}_1$

[43] You should verify that if $\varepsilon \leq 0.5$, then $\bar{r} \geq 0$ and the only equilibrium with $m > 0$ is the autarkic equilibrium in which money has no value. All other possible equilibrium paths eventually violate nonnegativity of consumption.

[44] But only in the limit. It is crucial that the real value of money is not zero at finite t, since with perfect foresight as in this model generation t would anticipate the fact that money would lose all its value, would not accept it from generation $t - 1$ and all monetary equilibria would unravel, with only the autarkic euqilibrium surviving.

[45] Generically in this context means, for almost all endowments, i.e. the set of possible values for the endowments for which this statement does not hold is of measure zero. Local uniquenes means that in for every equilibrium price vector there exists ε such that any ε-neighborhood of the price vector does not contain another equilibrium price vector (apart from the trivial ones involving a different normalization).

[46] Again we require the assumption that consumption in the first and the second period are strict gross substitutes, ruling out backward-bending offer curves.

Pareto-dominates the equilibrium indexed by p_1. By the same token, the *only* Pareto optimal equilibrium allocation is the nonautarkic stationary monetary equilibrium.

8.1.6 Productive Outside Assets

We have seen that with a positive supply of an outside asset with no intrinsic value, $m > 0$, then in the Samuelson case (for which the slope of the offer curve is smaller than one at the autarkic allocation) we have a continuum of equilibria. Now suppose that, instead of being endowed with intrinsically useless pieces of paper the initial old are endowed with a Lucas tree that yields dividends $d > 0$ in terms of the consumption good in each period. In a lot of ways this economy seems a lot like the previous one with money. So it should have the same number and types of equilibria!? The definition of equilibrium (we will focus on Arrow-Debreu equilibria) remains the same, apart from the resource constraint which now reads

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t + d$$

and the budget constraint of the initial old generation which now reads

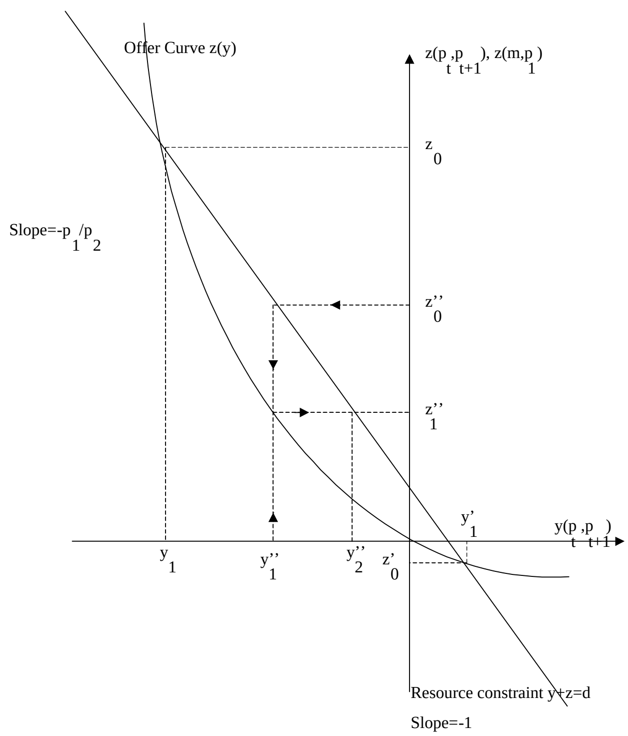
$$p_1 c_1^0 \leq p_1 e_1^0 + d \sum_{t=1}^{\infty} p_t$$

Let's analyze this economy using our standard techniques. The offer curve remains completely unchanged, but the resource line shifts to the right, now goes through the points $(y, z) = (1, 0)$ and $(y, z) = (0, 1)$. Let's look at Figure 11.

It appears that, as in the case with money $m > 0$ there are two stationary and a continuum of nonstationary equilibria. The point (y_1, z_0) on the offer curve indeed represents a stationary equilibrium. Note that the constant equilibrium price ratio satisfies $\frac{p_t}{p_{t+1}} = \alpha > 1$ (just draw a ray through the origin and the point and compare with the slope of the resource constraint which is -1). Hence we have, after normalization of $p_1 = 1$, $p_t = \left(\frac{1}{\alpha}\right)^{t-1}$ and therefore the value of the Lucas tree in the first period equals

$$d \sum_{t=1}^{\infty} \left(\frac{1}{\alpha}\right)^{t-1} < \infty$$

How about the other intersection of the resource line with the offer curve, (y_1', z_0')? Note that in this hypothetical stationary equilibrium $\frac{p_t}{p_{t+1}} = \gamma < 1$, so that $p_t = \left(\frac{1}{\gamma}\right)^{t-1} p_1$. Hence the period 0 value of the Lucas tree is infinite and the consumption of the initial old exceed the resources available in the economy in period 1. This obviously cannot be an equilibrium. Similarly all equilibrium paths starting at some point z_0'' converge to this stationary point, so for all hypothetical nonstationary equilibria we have $\frac{p_t}{p_{t+1}} < 1$ for t large enough and



Offer Curve z(y)
z(p t ,p t+1), z(m,p 1)
z 0
Slope=-p 1 /p 2
z'' 0
z'' 1
y' 1
y(p t ,p t+1)
y 1
y'' 1
y'' 2
z' 0
Resource constraint y+z=d
Slope=-1

Figure 11:

again the value of the Lucas tree remains unbounded, and these paths cannot be equilibrium paths either. We conclude that in this economy there exists a unique equilibrium, which, by the way, is Pareto optimal.

This example demonstrates that it is not the existence of a long-lived outside asset that is responsible for the existence of a continuum of equilibria. What is the difference? In all monetary equilibria apart from the stationary nonautarkic equilibrium (which exists for the Lucas tree economy, too) the price level goes to infinity, as in the hypothetical Lucas tree equilibria that turned out not to be equilibria. What is crucial is that money is intrinsically useless and does not generate real stuff so that it is possible in equilibrium that prices explode, but the real value of the dividends remains bounded. Also note that we were to introduce a Lucas tree with negative dividends (the initial old generation is an eternal slave, say, of the government and has to come up with d in every period to be used for government consumption), then the existence of the whole continuum of equilibria is restored.

8.1.7 Endogenous Cycles

Not only is there a possibility of a continuum of equilibria in the basic OLG-model, but these equilibria need not take the monotonic form described above. Instead, equilibria with cycles are possible. In Figure 12 we have drawn an offer curve that is backward bending. In the homework you will see an example of preferences that yields such a backward bending offer curve, for a rather normal utility function.

Let $m > 0$ and let p_1 be such that $z_0 = \frac{m}{p_1}$. Using our geometric approach we find $y_1 = y(p_1, p_2)$ from the resource line, $z_1 = z(p_1, p_2)$ from the offer curve (ignore for the moment the fact that there are several z_1 will do; this merely indicates that the multiplicity of equilibria is of even higher order than previously demonstrated). From the resource line we find $y_2 = y(p_2, p_3)$ and from the offer curve $z_2 = z(p_2, p_3) = z_0$. After period $t = 2$ the economy repeats the cycle from the first two periods. The equilibrium allocation is of the form

$$\begin{aligned} c_t^{t-1} &= \begin{cases} c^{ol} = z_0 - w_2 \text{ for } t \text{ odd} \\ c^{oh} = z_1 - w_2 \text{ for } t \text{ even} \end{cases} \\ c_t^t &= \begin{cases} c^{yl} = y_1 - w_1 \text{ for } t \text{ odd} \\ c^{yh} = y_2 - w_1 \text{ for } t \text{ even} \end{cases} \end{aligned}$$

with $c^{ol} < c^{oh}$, $c^{yl} < c^{yh}$. Prices satisfy

$$\begin{aligned} \frac{p_t}{p_{t+1}} &= \begin{cases} \alpha^h \text{ for } t \text{ odd} \\ \alpha^l \text{ for } t \text{ even} \end{cases} \\ \pi_{t+1} &= -r_{t+1} = \begin{cases} \pi^l < 0 \text{ for } t \text{ odd} \\ \pi^h > 0 \text{ for } t \text{ even} \end{cases} \end{aligned}$$

Consumption of generations fluctuates in a two period cycle, with odd generations eating little when young and a lot when old and even generations having

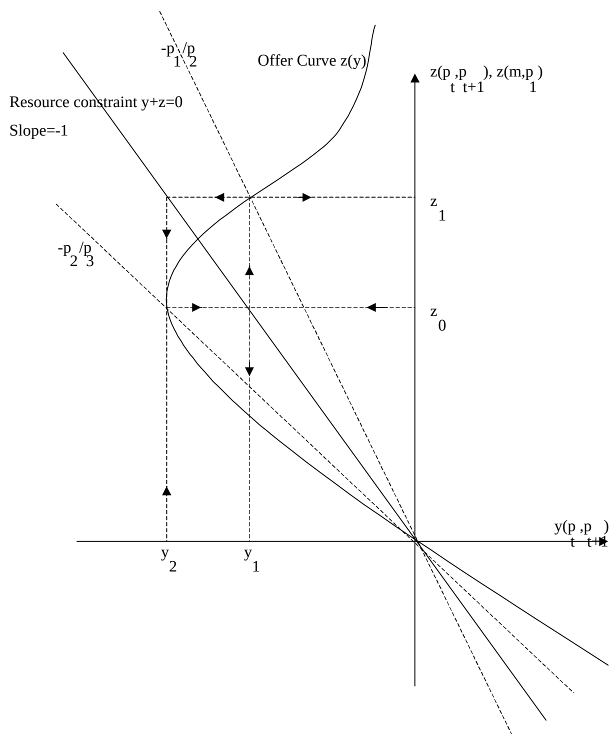

-p1/p2
Offer Curve z(y)
z(pt,pt+1), z(m,p1)
Resource constraint y+z=0
Slope=-1
z1
-p2/p3
z0
y(pt,pt+1)
y2
y1

Figure 12:

the reverse pattern. Equilibrium returns on money (inflation rates) fluctuate, too, with returns from odd to even periods being high (low inflation) and returns being low (high inflation) from even to odd periods. Note that these cycles are purely endogenous in the sense that the environment is completely stationary: nothing distinguishes odd and even periods in terms of endowments, preferences of people alive or the number of people. It is not surprising that some economists have taken this feature of OLG models to be the basis of a theory of endogenous business cycles (see, for example, Grandmont (1985)). Also note that it is not particularly difficult to construct cycles of length bigger than 2 periods.

8.1.8 Social Security and Population Growth

The pure exchange OLG model renders itself nicely to a discussion of a pay-as-you-go social security system. It also prepares us for the more complicated discussion of the same issue once we have introduced capital accumulation. Consider the simple model without money (i.e. $m = 0$). Also now assume that the population is growing at constant rate n, so that for each old person in a given period there are $(1+n)$ young people around. Definitions of equilibria remain unchanged, apart from resource feasibility that now reads

$$c_t^{t-1} + (1+n)c_t^t = e_t^{t-1} + (1+n)e_t^t$$

or, in terms of excess demands

$$z(p_{t-1}, p_t) + (1+n)y(p_t, p_{t+1}) = 0$$

This economy can be analyzed in exactly the same way as before with noticing that in our offer curve diagram the slope of the resource line is not -1 anymore, but $-(1+n)$. We know from above that, without any government intervention, the unique equilibrium in this case is the autarkic equilibrium. We now want to analyze under what conditions the introduction of a pay-as-you-go social security system in period 1 (or any other date) is welfare-improving. We again assume stationary endowments $e_t^t = w_1$ and $e_{t+1}^t = w_2$ for all t. The social security system is modeled as follows: the young pay social security taxes of $\tau \in [0, w_1)$ and receive social security benefits b when old. We assume that the social security system balances its budget in each period, so that benefits are given by

$$b = \tau(1+n)$$

Obviously the new unique competitive equilibrium is again autarkic with endowments $(w_1 - \tau, w_2 + \tau(1+n))$ and equilibrium interest rates satisfy

$$1 + r_{t+1} = 1 + r = \frac{U'(w_1 - \tau)}{\beta U'(w_2 + \tau(1+n))}$$

Obviously for any $\tau > 0$, the initial old generation receives a windfall transfer of $\tau(1+n) > 0$ and hence unambiguously benefits from the introduction. For all other generations, define the equilibrium lifetime utility, as a function of the social security system, as

$$V(\tau) = U(w_1 - \tau) + \beta U(w_2 + \tau(1+n))$$

The introduction of a small social security system is welfare improving if and only if $V'(\tau)$, evaluated at $\tau = 0$, is positive. But

$$\begin{aligned} V'(\tau) &= -U'(w_1 - \tau) + \beta U'(w_2 + \tau(1+n))(1+n) \\ V'(0) &= -U'(w_1) + \beta U'(w_2)(1+n) \end{aligned}$$

Hence $V'(0) > 0$ if and only if

$$n > \frac{U'(w_1)}{\beta U'(w_2)} - 1 = \bar{r}$$

where $\bar{r}$ is the autarkic interest rate. Hence the introduction of a (marginal) pay-as-you-go social security system is welfare improving if and only if the population growth rate exceeds the equilibrium (autarkic) interest rate, or, to use our previous terminology, if we are in the Samuelson case where autarky is not a Pareto optimal allocation. Note that social security has the same function as money in our economy: it is a social institution that transfers resources between generations (backward in time) that do not trade among each other in equilibrium. In enhancing intergenerational exchange not provided by the market it may generate allocations that are Pareto superior to the autarkic allocation, in the case in which individuals private marginal rate of substitution $1 + \bar{r}$ (at the autarkic allocation) falls short of the social intertemporal rate of transformation $1 + n$.

If $n > \bar{r}$ we can solve for optimal sizes of the social security system analytically in special cases. Remember that for the case with positive money supply $m > 0$ but no social security system the unique Pareto optimal allocation was the nonautarkic stationary allocation. Using similar arguments we can show that the sizes of the social security system for which the resulting equilibrium allocation is Pareto optimal is such that the resulting autarkic equilibrium interest rate is at least equal to the population growth rate, or

$$1 + n \leq \frac{U'(w_1 - \tau)}{\beta U'(w_2 + \tau(1+n))}$$

For the case in which the period utility function is of logarithmic form this yields

$$\begin{aligned} 1 + n &\leq \frac{w_2 + \tau(1+n)}{\beta(w_1 - \tau)} \\ \tau &\geq \frac{\beta}{1+\beta} w_1 - \frac{w_2}{(1+\beta)(1+n)} = \tau^*(w_1, w_2, n, \beta) \end{aligned}$$

Note that τ^* is the unique size of the social security system that maximizes the lifetime utility of the representative generation. For any smaller size we could marginally increase the size and make the representative generation better off and increase the windfall transfers to the initial old. Note, however, that any $\tau > \tau^*$ satisfying $\tau \leq w_1$ generates a Pareto optimal allocation, too: the representative generation would be better off with a smaller system, but the initial old generation would be worse off. This again demonstrates the weak requirements that Pareto optimality puts on an allocation. Also note that the "optimal" size of social security is an increasing function of first period income w_1, the population growth rate n and the time discount factor β, and a decreasing function of the second period income w_2.

So far we have assumed that the government sustains the social security system by forcing people to participate.[47] Now we briefly describe how such a system may come about if policy is determined endogenously. We make the following assumptions. The initial old people can decide upon the size of the social security system $\tau_0 = \tau^{**} \geq 0$. In each period $t \geq 1$ there is a majority vote as to whether the current system is to be kept or abolished. If the majority of the population in period t favors the abolishment of the system, then $\tau_t = 0$ and no payroll taxes or social security benefits are paid. If the vote is in favor of the system, then the young pay taxes τ^{**} and the old receive $(1+n)\tau^{**}$. We assume that $n > 0$, so the current young generation determines current policy. Since current voting behavior depends on expectations about voting behavior of future generations we have to specify how expectations about the voting behavior of future generations is determined. We assume the following expectations mechanism (see Cooley and Soares (1999) for a more detailed discussion of justifications as well as shortcomings for this specification of forming expectations):

$$\tau_{t+1}^e = \begin{cases} \tau^{**} \text{ if } \tau_t = \tau^{**} \\ 0 \text{ otherwise} \end{cases} \tag{56}$$

that is, if young individuals at period t voted down the original social security system then they expect that a newly proposed social security system will be voted down tomorrow. Expectations are rational if $\tau_t^e = \tau_t$ for all t. Let $\tau = \{\tau_t\}_{t=0}^{\infty}$ be an arbitrary sequence of policies that is feasible (i.e. satisfies $\tau_t \in [0, w_1)$)

Definition 80 *A rational expectations politico-economic equilibrium, given our expectations mechanism is an allocation rule* $\hat{c}_1^0(\tau), \{(\hat{c}_t^t(\tau), \hat{c}_{t+1}^t(\tau))\}$, *price rule* $\{\hat{p}_t(\tau)\}$ *and policies* $\{\hat{\tau}_t\}$ *such that*[48]

1. *for all* $t \geq 1$, *for all feasible* τ, *and given* $\{\hat{p}_t(\tau)\}$,

$$\begin{aligned} (\hat{c}_t^t, \hat{c}_{t+1}^t) \quad &\in \quad \arg\max_{(c_t^t, c_{t+1}^t) \geq 0} V(\tau_t, \tau_{t+1}) = U(c_t^t) + \beta U(c_{t+1}^t) \\ s.t.\ p_t c_t^t + p_{t+1} c_{t+1}^t \quad &\leq \quad p_t (w_1 - \tau_t) + p_{t+1} (w_2 + (1+n)\tau_{t+1}) \end{aligned}$$

[47] This section is not based on any reference, but rather my own thoughts. Please be aware of this and read with caution.

[48] The dependence of allocations and prices on τ is implicit from now on.

2. *for all feasible* τ, *and given* $\{\hat{p}_t(\tau)\}$,

$$\begin{aligned} \hat{c}_1^0 &\in \arg\max_{c_1^0 \geq 0} V(\tau_0, \tau_1) = U(c_1^0) \\ s.t.\ p_1 c_1^0 &\leq p_1(w_2 + (1+n)\tau_1) \end{aligned}$$

3.

$$c_t^{t-1} + (1+n)c_t^t = w_2 + (1+n)w_1$$

4. *For all* $t \geq 1$

$$\hat{\tau}_t \in \arg \max_{\theta \in \{0, \tau^{**}\}} V(\theta, \tau_{t+1}^e)$$

where τ_{t+1}^e *is determined according to* (56)

5.

$$\hat{\tau}_0 \in \arg \max_{\theta \in [0, w_1)} V(\theta, \hat{\tau}_1)$$

6. *For all* $t \geq 1$

$$\tau_t^e = \hat{\tau}_t$$

Conditions 1-3 are the standard economic equilibrium conditions for any arbitrary sequence of social security taxes. Condition 4 says that all agents of generation $t \geq 1$ vote rationally and sincerely, given the expectations mechanism specified. Condition 5 says that the initial old generation implements the best possible social security system (for themselves). Note the constraint that the initial generation faces in its maximization: if it picks θ too high, the first regular generation (see condition 4) may find it in its interest to vote the system down. Finally the last condition requires rational expectations with respect to the formation of policy expectations.

Political equilibria are in general very hard to solve unless one makes the economic equilibrium problem easy, assumes simple voting rules and simplifies as much as possible the expectations formation process. I tried to do all of the above for our discussion. So let find an (the!) political economic equilibrium. First notice that for any policy the equilibrium allocation will be autarky since there is no outside asset. Hence we have as equilibrium allocations and prices for a given policy τ

$$\begin{aligned} c_t^{t-1} &= w_2 + (1+n)\tau_t \\ c_t^t &= w_1 - \tau_t \\ p_1 &= 1 \\ \frac{p_t}{p_{t+1}} &= \frac{U'(w_1 - \tau_t)}{\beta U'(w_2 + (1+n)\tau_t)} \end{aligned}$$

Therefore the only equilibrium element to determine are the optimal policies. Given our expectations mechanism for any choice of $\tau_0 = \tau^{**}$, when would generation t vote the system τ^{**} down when young? If it does, given the expectation mechanism, it would not receive benefits when old (a newly installed system would be voted down right away, according to the generations' expectation). Hence

$$V(0, \tau_{t+1}^e) = V(0, 0) = U(w_1) + \beta U(w_2)$$

Voting to keep the system in place yields

$$V(\tau^{**}, \tau_{t+1}^e) = V(\tau^{**}, \tau^{**}) = U(w_1 - \tau^{**}) + \beta U(w_2 + (1+n)\tau^{**})$$

and a vote in favor requires

$$V(\tau^{**}, \tau^{**}) \geq V(0, 0) \tag{57}$$

But this is true for all generations, including the first regular generation. Given the assumption that we are in the Samuelson case with $n > \bar{r}$ there exists a $\tau^{**} > 0$ such that the above inequality holds. Hence the initial old generation can introduce a positive social security system with $\tau_0 = \tau^{**} > 0$ that is not voted down by the next generation (and hence by no generation) and creates positive transfers for itself. Obviously, then, the optimal choice is to maximize $\tau_0 = \tau^{**}$ subject to (57), and the equilibrium sequence of policies satisfies $\hat{\tau}_t = \tau^{**}$ where $\tau^{**} > 0$ satisfies

$$U(w_1 - \tau^{**}) + \beta U(w_2 + (1+n)\tau^{**}) = U(w_1) + \beta U(w_2)$$