A Note on Betas, Leverage and Taxes

One issue that is often confusing is the relation between the asset β (called β_A, or β_U) and the equity β (β_E). This relation is a function of the firm's leverage, the tax rate, and—in a point that is often missed—the firm's leverage *policy*, how the firm will change is leverage rate over time.

First, take the simple case of a no tax world. In that case it is easy to see what the relation is. We can go about it in two ways. First, start with the Modigliani-Miller proposition on the relation between the leverage and the cost of equity capital. From M&M we know that in a no-tax world:

$$k_E = k_U + \left[\frac{D}{E}\right]\left[k_U - k_D\right] \tag{1}$$

where k_U is the unlevered cost of capital for the firm, k_E is the (levered) cost of equity capital, k_D is the cost of debt and D and E are the market value of debt and equity. Now, if we make the further assumption that CAPM holds, which is useful, but not necessary for M-M to hold, we know that because the various costs of capital are in fact the rates of return that investors would expect to receive on the corresponding investments, we get:

$$\begin{aligned} k_E &= E(R_E) = R_F + b_E\left[E(R_M) - R_F\right] \\ k_U &= E(R_U) = R_F + b_U\left[E(R_M) - R_F\right] \end{aligned} \tag{2}$$

If we further assume that β_D=0, we know that k_D=R_F. Now, if we substitute from equation (2) into equation (1), we get:

$$\begin{aligned} R_F + b_E\left[E(R_M) - R_F\right] &= R_F + b_U\left[E(R_M) - R_F\right] + \left[\frac{D}{E}\right]\left[R_F + b_U\left[E(R_M) - R_F\right] - R_F\right] \\ b_E\left[E(R_M) - R_F\right] &= b_U\left[E(R_M) - R_F\right] + \left[\frac{D}{E}\right]\left[b_U\left[E(R_M) - R_F\right]\right] \\ b_E &= \left[1 + \left(\frac{D}{E}\right)\right] b_U \\ &= \left[1 + \left(\frac{D}{E}\right)\right] b_A \end{aligned} \tag{3}$$

Note that if D=0 then β_E=β_U and the systematic risk of the equity depends only on the risk of the firm's assets. This is why β_U=β_A. Another way to see that this must be true is to consider the firm's balance sheet. Imagine a balance sheet that is computed using the market value of the firm's assets and liabilities.[1]

[1] You also need to recall that if you have two assets, the β of the portfolio is simply the weighted sum of the individual asset β's, where the weights are proportional to the market values of the assets. So, if I have

Assets		**Liabilities and Owner's Equity**	
Total Assets ($\beta=\beta_A$)	A	D	Liabilities ($\beta=\beta_D$)
		S	Equity ($\beta=\beta_E$)
Assets ($\beta=\beta_A$)	A	S+D=V	Liabilities + O.E. ($\beta=\beta_V$)

Since the left side of the B/S is always equal to the right side, it must be the case that the β of the (Liabilities + Owner's Equity) is equal to the β of the Assets. So,

$$\beta_A = \beta_V \Rightarrow$$
$$\beta_A = \left[\frac{E}{V}\right]\beta_E + \left[\frac{D}{V}\right]\beta_D \tag{4}$$

Let $\beta_D=0$ and simplify, obtaining:

$$\beta_A = \left[\frac{E}{V}\right]\beta_E$$
$$\beta_E = \left[\frac{V}{E}\right]\beta_A \tag{5}$$
$$= \left[1+\left(\frac{D}{E}\right)\right]\beta_A$$

which is the same as equation (3) when we recognize that $\beta_A \equiv \beta_U$.

Now, we can modify things and assume that there is a corporate income tax at the rate τ, with interest being deductible from taxable income. Further, assume that the dollar amount of debt that the firm has outstanding is constant (or at least known with certainty).

In this case we know from M&M that the value of the levered firm exceeds the value of an otherwise identical unlevered firm by τD, the present value of the tax shield associated with a constant amount of debt, D. That is,

$$E + D = V_L = V_U + \tau D \tag{6}$$

To see what happens next, we can return to our balance sheet, but modify it for the presence of the tax shield:

$750 invested in GM (which has a β of .5) and $250 invested in IBM (which has a β of 1.5), the β of the portfolio is .75 x .5 + .25 x 1.5 = .75.

Assets		**Liabilities and Owner's Equity**	
Total Physical Assets ($\beta=\beta_a$)	A	D	Liabilities ($\beta=\beta_D$)
Tax Shield	τD	E	Equity ($\beta=\beta_E$)
Assets	A+τD	E+D=V_L	Liabilities + O.E.

Since the β's of two investments that have the same value at each point in time must be the same, we get:

$$\beta_E\left[\frac{E}{V_L}\right]+\beta_D\left[\frac{D}{V_L}\right]=\beta_A\left[\frac{V_U}{V_L}\right]+\left[\frac{\tau D}{V_L}\right]\beta_D \tag{7}$$

Since β_D=0, we get:

$$\beta_E\left[\frac{E}{V_L}\right]=\beta_A\left[\frac{V_U}{V_L}\right] \tag{8}$$

Now we can also get the standard result for the relation between β_E and β_A when there are corporate taxes. The logic is simply that the firm has gained a new asset, the tax shield, and this new asset has a β of zero as the amount of debt will not vary in any systematic way with the of the market. We can see this if we return to our Balance Sheet.

Now assume that the β of the tax shield, β_{TS} is=0. We also know that V_U=A, the value of the unlevered version of our firm is the value of its assets. Again, assume that β_D=0.

$$\begin{aligned}\left[\frac{E}{V_L}\right]\beta_E+\left[\frac{D}{V_L}\right]\beta_D&=\left[\frac{A}{V_L}\right]\beta_A+\left[\frac{\tau D}{V_L}\right]\beta_{TS}\\ \beta_E E&=\beta_A A\\ &=\beta_A\left[V_L-\tau D\right]\\ &=\beta_A\left[E+D-\tau D\right]\\ \beta_E&=\left[\frac{E+(1-\tau)D}{E}\right]\beta_A\\ &=\left[1+(1-\tau)\left(\frac{D}{E}\right)\right]\beta_A\end{aligned} \tag{9}$$

In equation (9) the step from the first line to the second line simply applies the assumption that the tax-shield and debt β's are both 0. The last line of equation (9) is just the formula for the relation that you see in the Ross, Westerfield and Jaffe text.

However, the relation in the last line of equation (9) depends crucially on the assumption that β_{TS}=0. That is, it depends on an assumption about the capital structure *policy* of the firm, that the debt will in constant dollar amount. Thus, this assumed relation between asset and equity β's will not hold true for all capital structure policies. In particular, look what happens when we make what may be a more reasonable assumption, that the firm will vary its capital structure to keep a constant (in market value terms) leverage ratio. In that case, it would seem that the appropriate value for β_{TS} is β_A. A firm that keeps a constant percentage of its value in debt will find that the face value of its debt also varies with the market in the same way as the value of the assets. Thus, if the firms asset β is β_A, then β_{TS}=β_A. Now we can go through a similar analysis to that presented in equation (9).

$$
\begin{aligned}
\left[\frac{E}{V_L}\right] b_E + \left[\frac{D}{V_L}\right] b_D &= \left[\frac{A}{V_L}\right] b_A + \left[\frac{tD}{V_L}\right] b_{TS} \\
&= \left[\frac{A}{V_L}\right] b_A + \left[\frac{tD}{V_L}\right] b_A \\
&= \left[\frac{A + tD}{V_L}\right] b_A \qquad (10) \\
&= \left[\frac{V_L}{V_L}\right] b_A \\
&= b_A
\end{aligned}
$$

Now, all we have to do is multiply both sides by V_L and we get:

$$
\begin{aligned}
b_E E &= b_A V_L \\
b_E &= b_A \left[\frac{V_L}{E}\right] \\
&= \left[\frac{E + D}{E}\right] b_A \qquad (11) \\
&= \left[1 + \left(\frac{D}{E}\right)\right] b_A
\end{aligned}
$$

Note that the last line of equation (11) is identical to the last line of equation (5). That is, if the firm is going to maintain a constant *proportion* of its capital structure in debt, the tax adjustment in the relation between β_E and β_A is not needed. If the firm is going to maintain a constant *amount* of debt in its capital structure, the adjustment is needed.