

Optimal Acquisition of a Partially Hedgeable House

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Abstract

We consider the problem of the optimal time to purchase a house by a risk-averse investor who has access to complete financial markets and whose objective is to maximize expected utility from wealth at some fixed horizon. The house purchase is financially attractive (due to tax advantages, for example), which provides an incentive to buy as soon as possible; however, its value is only partially correlated with financial markets and, therefore, house price risk cannot be perfectly hedged, which provides an incentive to delay purchase. We are able to fully characterize the problem in the case of CARA utility, so that we can generate simple numerical solutions for different parameter values, and study the trade-off between the two conflicting incentives.

Keywords: Stopping Time, Incomplete Markets, Martingale Methods

JEL classification: C61, D81, G11.

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1 Introduction

The main financial vehicle to save/invest wealth for US households is real estate, namely the house where the household will reside.¹ There are strong tax incentives for households to purchase a house. The majority of households will purchase a house at some point, borrowing part of the price through a mortgage. Throughout the years, a proportion of the regular income of the household will be allocated to the payment of the mortgage. The equity accumulated in the house will represent savings that can be used to finance the lifestyle of the parents after retirement: For example, they might move to a smaller place and monetize part of the equity, or even get a reverse mortgage in the house, to use up the equity accumulated. In addition, it represents a financial buffer in case of some unforeseen emergency, as sudden unemployment, for example. On the other hand, house ownership involves financial risks as well. In particular, real estate prices are stochastic and subject to economic cycles, sometimes with long-lasting negative trends. Furthermore, transaction costs are very high and fully hedging real estate prices, at least for individual investors, does not seem feasible.² However, and despite the importance that the house purchase decision has among all economic decisions of households, the finance literature on optimal investment strategies has traditionally ignored both the housing purchase decision, as well as the interaction between ownership of real estate and optimal portfolio allocation.

In this paper we consider the problem of an agent who has a one-time opportunity, within a finite time horizon, to buy a house with a stochastic price. The agent starts with a given level of wealth that can be allocated across different financial securities in a complete financial markets setting. Besides, the agent has the opportunity to invest part of the wealth in a house, that we assume to be completely illiquid until the end of the time horizon, and whose prices is only partially correlated with the stock market, and therefore, wealth invested in the house cannot be perfectly hedged. On the other hand, there is an incentive to buy the house; in particular, we assume that the house purchase provides a lump-sum addition to total wealth due to, for example, the present value of tax benefits. The total value of the incentive is decreasing in time,

¹That is the case also in other countries, especially developed countries, but the incentives vary, that is why for motivation purposes we focus our discussion on the US.

²For example, Piazzesi, Schneider and Tuzel (2007) report a very low correlation -0.05- between stock market returns and returns on a widely used index of house prices.

which provides an incentive to buy early. However, the illiquid nature of the investment provides an incentive to delay purchase, until the relative value of the house with respect to accumulated wealth is not very high. We further assume that the agent cares about utility of final total (real estate plus financial) wealth. The end of the fixed time horizon might be interpreted as the time at which the agent decides to retire and/or start consuming the wealth saved in the house. We study how the decision to purchase depends on different parameter values, especially risk-aversion of the agent and correlation between the price of the house and the price of the risky security.

One of the first papers to consider the interaction between housing (or illiquid goods) ownership and other investment decisions is Grossman and Laroque (1990). They derive equilibrium implications from a representative agent who derives utility from ownership of an illiquid good (illiquid because its sale involves a transaction cost) and has to decide how to invest the rest of the wealth, and when to trade this good for another durable good; in their model, the value of the durable good depreciates at a constant rate. Flavin and Yamashita (2002) extend the previous model by allowing for consumption of goods other than the durable good and find a large number of general equilibrium implications. Cocco (2005) calibrates the problem of an investor who chooses consumption, level of housing and optimal portfolio allocation in a setting in which house prices, income and security prices are partially correlated; the existence of transaction costs makes housing illiquid; house ownership prevents young and poor investors from participating in equity markets. Cauley, Pavlov and Schwartz (2007) consider the optimal portfolio allocation problem of an investor who is already a homeowner and find the welfare impact of the housing constraint. A paper closely related is Miao and Wang (2007), who consider the optimal purchase decision of an investor who has access to security markets partially correlated with the price of the asset; the cost of the asset is fixed (as a strike price) but not its price; in their model, the objective of the investor is to maximize utility from intertemporal consumption and the asset can be consumed, therefore their problem is tantamount to finding the optimal exercise policy of the real option. Finally, Tebaldi and Schwartz (2007) solve the problem of optimal consumption and portfolio allocation when the investor has an illiquid asset.

From a technical standpoint, related papers are Cvitanić and Karatzas (1992) on optimal

investment allocation with incomplete markets, Karatzas and Wang (2001), who characterize the solution of mixed optimal stopping and control problems (as the one we consider in this paper); finally, the papers by Brendle and Carmona (2004), and Hugonnier and Morellec (2007), among others, consider the problem of hedging with incomplete markets.

In this paper, we consider a mixed portfolio allocation and optimal stopping time problem of a risk-averse investor with a random income and financial wealth in a finite time horizon. The investor has to decide when to purchase a house that offers an immediate wealth reward, because it can be purchased at a price lower than its asset value. As in Miao and Wang (2007), we want to characterize the optimal stopping time at which the house purchase takes place. Unlike in Miao and Wang (2007), purchase of the house ties a part of the wealth of the investor in an illiquid asset that only can be partially hedged with the rest of the wealth of the investor (the financial wealth). More importantly, for the economic problem we consider we need a finite horizon -unlike Miao and Wang (2007), who assume infinite horizon. This makes our numerical problem substantially harder to solve. In particular, we are required to use a different methodology, (martingale duality). Despite market incompleteness -when we take in consideration the illiquid nature of the house-, we are able to provide a semi-analytic characterization of the optimal stopping time, perform numerical exercises through simulations and study the comparative statics of the problem.

The paper is structured as follows: In section 2 we describe our model; in section 3 we find the optimal portfolio allocation strategy and characterize the optimal stopping time; section 4 is devoted to a numerical application; in section 5 we consider the problem of selling the house and buying another one; we close the paper with some conclusions.

2 The Model

Let $W = [W^1 \quad W^2]'$ be a two dimensional standard Brownian motion process, and let the σ -algebras F , F^1 and F^2 be the usual augmented filtrations generated by W , W^1 and W^2 , respectively. All the model parameters are assumed to be adapted to F^1 on a probability space (Ω, F, P) . The set of square-integrable, F^1 -adapted processes over an interval $[s, t]$, will be denoted by $L_{F^1}^2[s, t]$. Similarly, the space of square-integrable random variables that are adapted

to F_T^1 will be denoted by $L_{F_T^1}^2(P)$. Our model consists of a (locally) risk-free asset with price S_0 , a risky stock S that represents the market portfolio, and a real estate investment (house) opportunity H . The details are given below:

- Risk-free asset: $\frac{dS_0}{S_0}(t) = r(t)dt$, where the risk-free rate $r(t)$ is bounded and adapted to F^1 .
- Stock price dynamics: $\frac{dS}{S}(t) = \mu(t)dt + \sigma(t)d\hat{W}(t)$, where $-1 < \rho < 1$, μ (instantaneous rate of return) and σ (volatility) are both adapted to F^1 , and $\hat{W} = \rho W^1 + \sqrt{1 - \rho^2}W^2$ is also a standard one-dimensional Brownian motion process.
- Financial wealth: With $X(0) = x_0$, the dynamics of the financial wealth process is

$$dX = \pi \frac{dS}{S} + (X - \pi) \frac{dS_0}{S_0} + I dt = [\pi(\mu - r) + rX + I]dt + \pi\sigma d\hat{W}, \quad (2.1)$$

where π is the amount of the wealth invested in the risky asset and I is the net (of the consumption) income rate of the investor, which is also adapted to the filtration F^1 and satisfies $\int_0^T |I(u)| du < \infty$, a.s (almost surely).

- House price dynamics: The price of the house in the market satisfies

$$\begin{aligned} dH &= H[\mu^H dt + \sigma^H dW^1] \\ H(0) &= h_0 > 0, \end{aligned}$$

with coefficients μ^H and σ^H essentially bounded and adapted to F^1 .³

- We assume that the parameters μ and μ^H are (essentially) bounded.
- We also assume that the volatility processes σ and σ^H are square-integrable (in $L^2(P)$ sense) and uniformly non-degenerate on $[0, T]$, a.s.

³Therefore, the coefficient ρ represents the correlation between the stock price process and the house price process. We assume they are less than perfectly correlated, as documented in the literature. As stated in the introduction, Piazzesi, Schneider and Tuzel (2007) report a of only 0.05 between stock market returns and returns on a widely used index of house prices.

- At some optimal time τ , $0 \leq \tau \leq T$, the investor decides to buy the house whose price is $H(\tau)$; however, the investor will only have to pay $\delta H(\tau)$, $0 < \delta(\tau) < 1$, where $(1 - \delta)H(\tau)$ accounts for the monetary value of the utility the investor derives from owning a house (including the value of tax and rent savings) at time $t = \tau$. Alternatively, if buying and using the house become more costly (e.g. higher insurance and property tax payments than tax savings, high renovation costs or lack of utility due to construction, noise, etc.) for some $t = \tau$, then $\delta(\tau) > 1$, and $(1 - \delta)H(\tau)$ would represent the monetary value of the disutility from buying the house. Therefore, $X(\tau) = X(\tau_-) - \delta H(\tau)$, and total terminal wealth $X^\pi(T) + H(T)$ can be written as

$$X(\tau) + H(\tau) + [\pi(\mu - r) + rX + I + H\mu^H]dt + \int_{\tau}^T [\pi\rho\sigma + H\sigma^H - \pi\sqrt{1 - \rho^2}\sigma] \begin{bmatrix} dW^1 \\ dW^2 \end{bmatrix}, \quad (2.2)$$

for any admissible portfolio π . We denote by Y the wealth of the investor, that is, $X + H$ after τ and X up to then.

- The objective of the investor is to maximize expected utility from the total terminal wealth: $E[u(Y(T))]$, where $u(x)$ is the exponential utility function: $u(x) = -e^{-\gamma x}$. We interpret T as the time at which the agent will start using the wealth accumulated because, for example, retirement. The monetization of the wealth invested in the house can take place through a sale or through other possible strategies, like reverse mortgages. For our purposes, we do not need to specify what happens after T , and it is enough to assume that at T the agent cares about the total value of wealth (financial and in real estate) and is risk-verse about it.

We discuss the solution to this problem in the next section.

3 Optimal Strategies

For $0 \leq s < t \leq T$, we define the set of admissible portfolio processes by $\mathcal{U}(s, t) = \{\pi \in L_{\mathcal{F}}^2[s, t] : \text{the equation (2.1) has a unique solution } X^\pi \in L_{\mathcal{F}}^2[s, t]\}$. Then the objective is to maximize $E[-e^{-\gamma Y(T)}]$ over all admissible pairs (τ, π) where τ is a stopping time with values between 0

and T , and is adapted to the filtration F . For each time t between 0 and T , the investor has an option to buy the house (the stopping time) so that the event $\{\tau \leq t\}$ holds, or wait and buy it at a future date, hoping that the expected utility will be higher. To make a decision, the investor should take the expected value of all the future scenarios into account. Assume that the current time is $t = 0$. We solve the investor's problem in a few steps: If τ is a candidate stopping time to purchase the house, we first solve the portfolio allocation problem

$$V^{\tau,x} = \sup_{\pi \in \mathcal{U}(\tau,T)} E_{\tau}^{\tau,x}[-e^{-\gamma Y(T)}], \quad (3.1)$$

after buying the house, where $E_{\tau}^{\tau,x}[\cdot]$ is the conditional expectation operator given F_{τ} , with $X(\tau) = x$. After that, we solve for the optimal portfolio before buying the house (before τ). Let V^{τ} denote the corresponding *objective* (or *reward*) function. Then V^{τ} satisfies

$$V^{\tau} = \sup_{\pi \in \mathcal{U}(0,\tau)} E[V^{\tau,X^{\pi}(\tau)}] = \sup_{\substack{\pi \in \mathcal{U}(0,T) \\ X(\tau) = X(\tau_-) - \delta H(\tau)}} E[-e^{-\gamma Y(T)}], \quad (3.2)$$

for fixed τ . At this stage, we can identify the best financial market allocation strategy and the resulting expected final utility for each stopping rule. The last part is then to decide which τ actually maximizes the objective function V^{τ} . The value function for the problem is then $V = \sup_{0 \leq \tau < T} V^{\tau}$, over all admissible τ .

3.1 Optimal Portfolio After Buying the House

In this section, we fix $\tau \in [0, T)$ and solve the optimization problem (3.1) with initial condition $X(\tau) = x$. For simplicity of the presentation, and of the proofs, we will assume that $r = 0$, however the results can be easily extended to the non-zero case by replacing the wealth process $X(t)$ with the discounted process $\bar{X}(t) = X(t)e^{-\int_{\tau}^t r(s)ds}$, and $\mu(\cdot)$ with $(\mu - r)(\cdot)$, as long as $r(\cdot)$ is bounded and adapted to F^1 .

We first introduce the following auxiliary process which is going to be fundamental for the

characterization of the optimal portfolios:

$$l(s, t) = e^{-a[H(t) + \int_s^t I(u) du] - \int_s^t b(u) dW^1(u) - \int_s^t c(u) du}, \quad \tau \leq s < t \leq T, \quad (3.3)$$

where the parameters $a, b(\cdot)$ and $c(\cdot)$ are $a = \gamma(1 - \rho^2), b(t) = \frac{\mu\rho}{\sigma}(t)$ and $c(t) = \frac{\mu^2}{2\sigma^2}(t)$.

Lemma 1. *The random variable $l(0, T)$ satisfies:*

(i) $l(0, T) \in L_{F^{-1}}^p(P)$, for any $p \geq 1$.

(ii) In particular, $l(0, T)$ is square-integrable and $E[l^2] \leq e^{2M(\gamma+T+MT)}$.

Proof. Let M be an upper bound for the (essentially) bounded processes $\int_0^t |I(u)| du, |b(t)|$ and $|c(t)|$ over the interval $[0, T]$. By writing the random variable $l(0, T)$ as

$$l = e^{-aH(T)} e^{-\int_0^T aI(u) du + \int_0^T c(u) du} e^{-\int_0^T b(u) dW^1(u)}$$

and noting that $0 \leq a \leq \gamma, H(\cdot) \geq 0$ and hence $|e^{-paH(T)}| \leq 1$ a.s. for any $p \geq 1$, we get

$$E[l^p] \leq E[e^{-p\int_0^T aI(u) du + \int_0^T c(u) du}] E[e^{-\int_0^T pb(u) dW^1(u)}] \quad (3.4)$$

The exponent $-p\int_0^T aI(u) du + \int_0^T c(u) du$ in the first expression on the right-hand side of equation (3.4) is bounded a.s. and it is easy to see that $E[e^{-p\int_0^T aI(u) du + \int_0^T c(u) du}] \leq e^{pM(\gamma+T)}$. Since b is bounded, the second expression in the expected value satisfies the Novikov condition and is also finite :

$$E[e^{-\int_0^T pb(u) dW^1(u)}] \leq E[e^{\int_0^T p^2 b^2(u) du}] \leq e^{p^2 M^2 T}.$$

Combining these bounds, we obtain $E[l^2] \leq e^{pM(\gamma+T+pMT)}$. Part (ii) follows with $p = 2$. \square

Lemma 2. *There exists a process $\phi \in L_{F^{-1}}^2[\tau, T]$ such that*

$$H(T) = \frac{1}{a} \left\{ \int_{\tau}^T (\phi - b)(t) dW^1(t) + \int_{\tau}^T \left(\frac{1}{2} |\phi(u)|^2 - c \right)(t) dt - \ln E_{\tau}[l(\tau, T)] \right\} - \int_{\tau}^T I(u) du, \quad a.s. \quad (3.5)$$

Proof. By Itô's Representation Theorem and Lemma 1, there is a (unique) progressively measurable process ϕ such that

$$l(\tau, T) = E_\tau[l(\tau, T)] \exp\left(-\int_\tau^T \phi(u) dW^1(u) - \frac{1}{2} \int_\tau^T |\phi(u)|^2 du\right).$$

Then, by taking the natural logarithm of both sides and using the equation (3.3), we get

$$\begin{aligned} \ln E_\tau[l(\tau, T)] &= \ln l(\tau, T) + \int_\tau^T \phi(u) dW^1(u) + \frac{1}{2} \int_\tau^T |\phi(u)|^2 du \\ &= -a[H(T) + \int_\tau^T I(u) du] + \int_\tau^T (\phi - b)(u) dW^1(u) + \int_\tau^T \left(\frac{1}{2}|\phi|^2 - c\right)(u) du. \end{aligned}$$

After rearranging the integrals and solving for H , the result follows. \square

We claim that the optimal portfolio on $[\tau, T]$ is given by

$$\pi^*(t) \triangleq \frac{\mu - \frac{\gamma\rho\sigma}{a}(\phi - b)}{\gamma\sigma^2}(t) = \frac{\mu - \rho\sigma\phi}{\gamma(1 - \rho^2)\sigma^2}(t) \quad (3.6)$$

which is a square-integrable process thanks to the assumptions on the parameters and Lemma 1. To this end, we define the following processes:

Definition 1. Let π^* be as in (3.6). Then

$$\tilde{Z}(\tau, t) \triangleq \exp\left(-\int_\tau^t \sigma^Z dW(u) - \frac{1}{2} \int_\tau^t |\sigma^Z(u)|^2 du\right), \quad t \leq \tau < T$$

is the exponential martingale process after buying the house where

$$\sigma^Z \triangleq [\sigma^{Z1} \quad \sigma^{Z2}] \triangleq \left[\frac{\mu - \gamma(1 - \rho^2)\sigma^2\pi^*}{\rho\sigma} \quad \gamma\sigma\sqrt{1 - \rho^2}\pi^* \right]. \quad (3.7)$$

We also introduce the expected value operator \tilde{E} under the equivalent probability measure \tilde{P} that is defined as $\tilde{P}(A) \triangleq E[1_A \tilde{Z}(T)]$ for any event $A \in \mathcal{F}_T$.

Lemma 3. Let $0 \leq \tau < T$ be given and $X(\tau) = x$ be fixed. Then, all of the following hold:

(i) The process $\{\tilde{Z}(\tau, t), \tau \leq t \leq T\}$ is a martingale under P

(ii) The process $\{\tilde{W}(t), \tau \leq t \leq T\}$ with $\tilde{W}(t) = W(t) + \int_{\tau}^t \sigma^Z(u)du$ is a Brownian motion under \tilde{P}

(iii) $X^\pi(t) - \int_{\tau}^t I(u)du$, with $\tau \leq t \leq T$, is a martingale under \tilde{P} for any admissible portfolio process π . In particular, $\tilde{E}_\tau^{\tau, x}[X^\pi(t) - \int_{\tau}^t I(u)du] = x$, for any $\tau \leq t \leq T$.

Proof. (i)-(ii) The diffusion coefficient σ^Z is a square integrable process as a linear function of π^* (3.7). Hence the proofs follow from the dynamics of \tilde{Z} and Girsanov theorem.

(iii) By Itô's rule,

$$d(X\tilde{Z}) = \tilde{Z}[\pi\rho\sigma - \sigma^{Z2}X - \pi\sigma\sqrt{1-\rho^2} - \sigma^{Z2}X]dW + \tilde{Z}\pi[\mu - \rho\sigma\sigma^{Z1} - \sigma\sqrt{1-\rho^2}\sigma^{Z2}]dt + \tilde{Z}Idt$$

where the drift term $[\mu - \rho\sigma\sigma^{Z1} - \sigma\sqrt{1-\rho^2}\sigma^{Z2}]$ vanishes by (3.7). Then, it is easy to see that

$$X(t)\tilde{Z}(\tau, t) - \int_{\tau}^t \tilde{Z}(\tau, u)I(u)du = x + \int_{\tau}^t \tilde{Z}(\tau, u)[\pi\rho\sigma - \sigma^{Z1}X - \pi\sigma\sqrt{1-\rho^2} - \sigma^{Z2}X](u)dW(u)$$

is a square integrable martingale with

$$\tilde{E}_\tau^{\tau, x}[X^\pi(t) - \int_{\tau}^t I(u)du] = E_\tau^{\tau, x}[X(t)\tilde{Z}(\tau, t) - \int_{\tau}^t \tilde{Z}(\tau, u)I(u)du] = x, \quad \tau \leq t \leq T.$$

□

Lemma 4. Let π^* be as in (3.6) for fixed $\tau \in [0, T)$. Then we have

$$X^{\pi^*}(T) + H(T) = X(\tau) - \frac{1}{\gamma(1-\rho^2)} \ln E_\tau[l(\tau, T)] - \frac{1}{\gamma} \ln \tilde{Z}(\tau, T), \quad a.s. \quad (3.8)$$

Proof. Let the parameters a , $b(\cdot)$ and $c(\cdot)$ be as before. By equation (2.1) and Lemma 2, $X^{\pi^*}(T) + H(T)$ can be written as

$$X(\tau) - \frac{\ln E_\tau[l(\tau, T)]}{a} + \int_{\tau}^T \left[\frac{\phi - b}{a} + \pi^* \rho\sigma - \sigma\pi^* \sqrt{1-\rho^2} \right] dW + \int_{\tau}^T \left(\frac{1}{2a} \phi^2 - \frac{c}{a} + \mu\pi^* \right) dt$$

which is equivalent to the right hand side of (3.8) if and only if

$$\int_{\tau}^T \left[\frac{\phi - b}{a} + \pi^* \rho \sigma - \sigma \pi^* \sqrt{1 - \rho^2} \right] dW + \int_{\tau}^T \left(\frac{\phi^2 - 2c}{2a} + \mu \pi^* \right) dt = \frac{1}{\gamma} \left\{ \int_{\tau}^T [\sigma^{Z1} - \sigma^{Z2}] dW + \int_{\tau}^T \frac{|\sigma^Z|^2}{2} du \right\}. \quad (3.9)$$

Writing $\phi = \frac{\mu - \gamma(1 - \rho^2)\sigma^2 \pi^*}{\rho \sigma}$ from (3.6), using (3.7) and comparing the drift and diffusion terms in both sides of the equation (3.9), the result follows. \square

Theorem 1. *The value function $V^{\tau, x}$ of the optimization problem (3.1) is given by*

$$V^{\tau, x} = -e^{-\gamma(x + \tilde{E}_{\tau}^{\tau, x}[H(T) + \int_{\tau}^T I(u) du] - \tilde{E}_{\tau}^{\tau, x}[\ln \tilde{Z}(\tau, T)]} = -e^{-\gamma x} E_{\tau}[l(\tau, T)]^{\frac{1}{1 - \rho^2}}, \quad (3.10)$$

for fixed (τ, x) , with $0 \leq \tau < T$ and $x > 0$. Moreover, $\pi^*(t) = \frac{\mu - \rho \sigma \phi}{\gamma(1 - \rho^2)\sigma^2}(t)$, $\tau < t \leq T$, is an optimal portfolio for this problem.

Proof. After adding $\int_{\tau}^T I(u) du$ to both sides of the equation (3.8) and multiplying both sides by $-\gamma$, we apply the martingale property in Lemma 3 (iii) to obtain

$$-\gamma \tilde{E}_{\tau}^{\tau, x} \left[H(T) + \int_{\tau}^T I(u) du \right] = \frac{1}{1 - \rho^2} \ln E_{\tau}[l(\tau, T)] + \tilde{E}_{\tau}^{\tau, x}[\ln \tilde{Z}(\tau, T)]. \quad (3.11)$$

Hence, for any $\pi \in \mathcal{U}(\tau, T)$, by a change of measure and Jensen's inequality,

$$\begin{aligned} E_{\tau}^{\tau, x} [e^{-\gamma[X^{\pi}(T) + H(T)]}] &= \tilde{E}_{\tau}^{\tau, x} [e^{-\gamma(X^{\pi}(T) + H(T)) - \ln \tilde{Z}(\tau, T)}] \\ &\geq e^{-\gamma \tilde{E}_{\tau}^{\tau, x} [X^{\pi}(T) + H(T)] - \tilde{E}_{\tau}^{\tau, x} [\ln \tilde{Z}(\tau, T)]} \\ &= e^{-\gamma(x + \tilde{E}_{\tau}^{\tau, x} [H(T) + \int_{\tau}^T I(u) du] - \tilde{E}_{\tau}^{\tau, x} [\ln \tilde{Z}(\tau, T)])} \\ &= e^{-\gamma x + \frac{1}{1 - \rho^2} \ln E_{\tau}[l(\tau, T)]}, \end{aligned}$$

so that $V^{\tau, x} \leq -e^{-\gamma x} E_{\tau}[l(\tau, T)]^{\frac{1}{1 - \rho^2}}$, $\forall \pi \in \mathcal{U}(\tau, T)$. On the other hand, this upper bound is

attained with $\pi = \pi^*$: Applying Lemma 3 with $X(\tau) = x$ and noting that $E_\tau^{\tau,x}[\tilde{Z}(\tau, T)] = 1$,

$$\begin{aligned} E_\tau^{\tau,x}[-e^{-\gamma[X^{\pi^*}(T)+H(T)]}] &= -E_\tau^{\tau,x}[e^{-\gamma x + \frac{1}{1-\rho^2} \ln E_\tau[L(\tau, T)]} \tilde{Z}(\tau, T)] \\ &= -e^{-\gamma x} E_\tau[L(\tau, T)]^{\frac{1}{1-\rho^2}} E_\tau^{\tau,x}[\tilde{Z}(\tau, T)] \\ &= -e^{-\gamma x} E_\tau[l(\tau, T)]^{\frac{1}{1-\rho^2}}. \end{aligned}$$

□

Corollary 1. *Assume that all the model parameters are deterministic. Then the process ϕ in the representation of the optimal portfolio $\pi^*(t) = \frac{\mu - \rho\sigma\phi}{\gamma(1-\rho^2)\sigma^2}(t)$ is given by*

$$\phi(t) = a\sigma^H(t)E_t[L(t, T)H(T)] + aE_t[l(t, T) \int_t^T \mathcal{D}_t I(u) du] + b(t)E_t[l(t, T)]. \quad (3.12)$$

Moreover, if the income rate process I is also deterministic, then

$$\phi(t) = a\sigma^H(t)E_t[l(t, T)H(T)] + b(t)E_t[l(t, T)]. \quad (3.13)$$

Proof. The random variable $l(\tau, T)$ can be written as

$$l(\tau, T) = E_\tau[l(\tau, T)] - \int_\tau^T l(\tau, u)\phi(u)dW^1(u) \quad (3.14)$$

where, by the Clark-Ocone formula,

$$-l(\tau, t)\phi(t) = E_t[\mathcal{D}_t L(\tau, T)], \quad (3.15)$$

and \mathcal{D} is the Malliavin derivative operator with respect to the Brownian motion process W^1 .

When the parameters are deterministic, $\mathcal{D}_t l(\tau, T)$ can be computed as

$$\begin{aligned}\mathcal{D}_t l(\tau, T) &= l(\tau, T) \left\{ -a\sigma^H(t)H(T) - a\mathcal{D}_t \int_{\tau}^T I(u)du - b(t) \right\} \\ &= -l(\tau, T) \left\{ a\sigma^H(t)H(T) + \int_t^T \mathcal{D}_t I(u)du + b(t) \right\}.\end{aligned}$$

Then, the result follows from (3.15). When $I(t)$ is deterministic, the Malliavin derivatives vanish and we get (3.13). \square

Remark 1. *The conditional expectations on the right-hand side of (3.12) cannot be solved explicitly but could be approximated numerically thanks to the conditional Gaussian distribution of time-discretized processes $l(t_i, t_{i+1})$ and $l(t_i, t_{i+1})H(t_{i+1})$ over any partition $\{\tau = t_0 < t_1 < \dots < t_n = T\}$ of $[\tau, T]$, for any fixed (deterministic) stopping time τ .*

3.2 Optimal Portfolio Before Buying the House

Now, we are going to discuss the optimization problem (3.2) of finding an optimal portfolio $\pi^* \in \mathcal{U}(0, \tau)$ for a fixed stopping time τ . Assuming that $X = X^{\pi^*}$ is the optimal process after τ , by Theorem 1 we have that:

$$\begin{aligned}V^\tau &= \sup_{\pi \in \mathcal{U}(0, \tau)} E[V^{\tau, X^\pi(\tau)}] \\ &= \sup_{\pi \in \mathcal{U}(0, \tau)} E[-e^{-\gamma X^\pi(\tau) + \frac{1}{1-\rho^2} \ln E_\tau[l(\tau, T)]}] \\ &= \sup_{\pi \in \mathcal{U}(0, \tau)} E[-e^{-\gamma\{X^\pi(\tau_-) - \delta H(\tau)\} + \frac{1}{1-\rho^2} \ln E_\tau[l(\tau, T)]}] \end{aligned} \tag{3.16}$$

where $l(\tau, T)$ depends on $H(T)$ through equations (3.8) or (3.11). Moreover, V^τ can be written as

$$V^\tau = \sup_{\pi \in \mathcal{U}(0, \tau)} E[-e^{-\gamma\{X^\pi(\tau) - \delta H(\tau) - \frac{1}{a} \ln E_\tau[l(\tau, T)]\}}] \tag{3.17}$$

after replacing $X^\pi(\tau_-)$ with $X^\pi(\tau)$ (with an abuse of the notation), for simplicity.

Remark 2. *Note that the expression $\gamma\delta H(\tau) + \frac{1}{1-\rho^2} \ln E_\tau[l(\tau, T)]$ in (3.16) is adapted to the filtration F^1 and, therefore, the stochastic optimization problem before τ can be solved using an*

approach like the one we used in the previous subsection.

Notation 1. We introduce the following processes:

$$\begin{aligned} D(\tau) &\triangleq -\delta H(\tau) - \frac{1}{a} \ln E_\tau[l(\tau, T)] \\ M(\tau) &\triangleq e^{-a[D(\tau) + \int_0^\tau I(u)du] - \int_0^\tau b(u)dW^1(u) - \int_0^\tau c(u)du} \end{aligned} \quad (3.18)$$

for $\tau \in [0, T)$, where a, b and c are as before.

Lemma 5. Let $\tau \in [0, T)$ be given. Then there exists a process $\psi \in L^2_{\mathcal{F}^{-1}}[0, \tau]$ such that $D(\tau)$ can be written as

$$D(\tau) + \int_0^\tau I(u)du = \frac{1}{a} \left\{ \int_0^\tau (\psi - b)(t)dW^1 + \int_0^\tau \left(\frac{1}{2}\psi^2 - c\right)(t)dt - \ln E[M(\tau)] \right\}. \quad (3.19)$$

Proof. By Itô's representation theorem, the process $M(\tau)$ can be written as

$$M(\tau) = E[M(\tau)] \exp\left(-\int_0^\tau \psi(u)dW^1(u) - \frac{1}{2} \int_0^\tau |\psi(u)|^2 du\right),$$

for some $\psi \in L^2_{\mathcal{F}^{-1}}[0, \tau]$. The rest is similar to the proof of Lemma 2. \square

Definition 2. Let ψ and π^* be as in Lemma 5 and (3.6), respectively. We then extend the definitions of the portfolio and exponential martingale processes in the previous subsection as follows:

$$\begin{aligned} \hat{\pi}(t) &\triangleq \begin{cases} \pi_*(t), & 0 \leq t < \tau \\ \pi^*(t), & \tau \leq t \leq T \end{cases} \\ \tilde{Z}(s, t) &\triangleq \exp\left(-\int_s^t \sigma^Z dW(u) - \frac{1}{2} \int_s^t |\sigma^Z|^2 du\right) \end{aligned}$$

for $0 \leq t \leq T$ where

$$\begin{aligned}\sigma^Z &= [\sigma^{Z1} \ \sigma^{Z2}] \triangleq \left[\frac{\mu - \gamma(1 - \rho^2)\sigma^2\hat{\pi}}{\rho\sigma} \quad \gamma\sqrt{1 - \rho^2}\sigma\hat{\pi} \right] \\ \pi_*(t) &\triangleq \frac{\mu - \rho\sigma\psi}{\gamma(1 - \rho^2)\sigma^2}(t);\end{aligned}\tag{3.20}$$

In particular, when $s = 0$ we write $\tilde{Z}(0, t) = \tilde{Z}(t)$.

Corollary 2. *When all the model parameters are deterministic, the process ψ in the representation of the optimal portfolio $\pi_*(t) = \frac{\mu - \rho\sigma\psi}{\gamma(1 - \rho^2)\sigma^2}(t)$ is given by $\psi(t) = -E_t[\mathcal{D}_t M(\tau)/M(t)$, where*

$$E_t[\mathcal{D}_t M(\tau)] = E_t[M(\tau)(a\delta\sigma^H(t)H(\tau) + \frac{\mathcal{D}_t E_\tau[l(\tau, T)]}{E_\tau[l(\tau, T)]} - a \int_t^\tau \mathcal{D}_t I(u)du - b(t))].\tag{3.21}$$

Proof. Again, by the Clark-Ocone formula, $\psi(t) = -E_t[\mathcal{D}_t M(\tau)/M(t)$ and

$$\begin{aligned}-E_t[\mathcal{D}_t M(\tau)] &= aE_t[M(\tau)(\mathcal{D}_t D(\tau) + \mathcal{D}_t \int_0^\tau I(u)du + b(t))] \\ &= aE_t[M(\tau)(-\delta\sigma^H(t)H(\tau) - \frac{\mathcal{D}_t E_\tau[l(\tau, T)]}{aE_\tau[l(\tau, T)]} + \int_t^\tau \mathcal{D}_t I(u)du + b(t))].\end{aligned}\tag{3.22}$$

□

Lemma 6. *Let π_* be as in (3.20), and $X^\pi(0) = x_0$. Then, we have*

$$X^{\pi_*}(\tau_-) + D(\tau) = x_0 - \frac{1}{\gamma(1 - \rho^2)} \ln E[M(\tau)] - \frac{1}{\gamma} \ln \tilde{Z}(\tau).$$

Proof. Using Lemma 5 and the notation of Definition 2, the proof is similar to that of Lemma 4. □

Proposition 1. *For any admissible τ , the value function V^τ is given by*

$$V^\tau = -e^{-\gamma(x_0 + \tilde{E} \int_0^\tau I(u)du + \tilde{E}[D(\tau)] - \tilde{E}[\ln \tilde{Z}(\tau)]}\tag{3.23}$$

$$= -e^{-\gamma x_0} (E[M(\tau)])^{\frac{1}{1 - \rho^2}},\tag{3.24}$$

and π_* , given in (3.20), is an optimal control in $\mathcal{U}(0, \tau)$ before buying the house.

Proof. By (3.16) and (3.18), we get

$$V^\tau = \sup_{\pi \in \mathcal{U}(0, \tau)} E[-e^{-\gamma(X^\pi(\tau_-) + D(\tau))}]$$

for which the right hand-side of (3.23) is obtained as an upper bound thanks to the martingale property of $X(t) - \int_0^t I(u)du$ and Jensen's inequality :

$$\begin{aligned} E[-e^{-\gamma(X^\pi(\tau_-) + D(\tau))}] &= -\tilde{E}[-e^{-\gamma(X^\pi(\tau_-) + D(\tau)) - \ln \tilde{Z}(\tau)}] \\ &\leq -e^{-\gamma \tilde{E}[X^\pi(\tau_-) + D(\tau)] - \tilde{E}[\ln \tilde{Z}(\tau)]} \\ &= -e^{-\gamma(x_0 + \tilde{E} \int_0^\tau I(u)du + \tilde{E}[D(\tau)]) - \tilde{E}[\ln \tilde{Z}(\tau)]} \end{aligned}$$

for any $\pi \in \mathcal{U}(0, \tau)$. From Lemma 6, we get

$$-\gamma(\tilde{E} \int_0^\tau I(u)du + \tilde{E}[D(\tau)]) - \tilde{E}[\ln \tilde{Z}(\tau)] = \frac{1}{1 - \rho^2} \ln E[M(\tau)]$$

which gives the equality (3.24). Moreover, when $\pi = \pi_*$, this upper bound $-e^{-\gamma x_0} (E[M(\tau)])^{\frac{1}{1 - \rho^2}}$ is obtained by Lemma 6. \square

Corollary 3. For each $\tau \in [0, T]$,

$$\begin{aligned} \sup_{\substack{\pi \in \mathcal{U}(0, T) \\ X(\tau) = X(\tau_-) - \delta H(\tau)}} E[e^{-\gamma Y^\pi(T)}] &= V^\tau \\ &= -e^{-\gamma x_0} (E[M(\tau)])^{\frac{1}{1 - \rho^2}} \end{aligned}$$

with optimal portfolio $\hat{\pi}(t)$, as in Definition 2.

Proof. For any $\pi \in \mathcal{U}(0, T)$,

$$\begin{aligned}
E[-e^{-\gamma Y^\pi(T)}] &= -E[\tilde{E}_\tau[e^{-\gamma X^\pi(T) - \gamma H(T) - \ln \tilde{Z}(\tau, T)}]] \\
&\leq -E[e^{-\gamma(X^\pi(\tau-) - \delta H(\tau)) + \frac{1}{1-\rho^2} \ln E_\tau[l(\tau, T)]}] \\
&= -\tilde{E}[e^{-\gamma[X^\pi(\tau-) + D(\tau)] - \ln \tilde{Z}(\tau)}] \\
&\leq -e^{-\gamma(x_0 + \tilde{E} \int_0^\tau I(u) du + \tilde{E}[D(\tau)] - \tilde{E}[\ln \tilde{Z}(\tau)]} = -e^{-\gamma x_0 + \frac{1}{1-\rho^2} \ln E[M(\tau)]}
\end{aligned}$$

which is clearly achieved by the portfolio $\hat{\pi}(t)$. □

3.3 Optimal Stopping Time

Then, the last part of this optimization problem is to solve the optimal stopping time problem:

$$\begin{aligned}
V &= \sup_{0 \leq \tau < T} V^\tau \\
&= -e^{-\gamma x_0} \inf_{0 \leq \tau < T} (E[M(\tau)])^{\frac{1}{1-\rho^2}}
\end{aligned} \tag{3.25}$$

which does not have an explicit analytical solution. However, we can simplify this problem, thanks to the monotonicity of the function $f(u) = u^{\frac{1}{1-\rho^2}}$, for any fixed ρ with $-1 < \rho < 1$.

Notation 2. For simplicity, we will consider the following Gaussian process for the net income rate: $I(t) = \mu^I(t) + \int_0^t \sigma^I(u) dW(u)$.

Corollary 4. Assume all the model parameters $\mu, \sigma, \mu^H, \sigma^H, \mu^I$ and σ^I are deterministic and let $\gamma > 0$ and $-1 < \rho < 1$ be fixed. Then the optimal time τ of (3.25) minimizes

$$E \left[e^{a[\delta H(\tau) - H(T) - \int_0^\tau I(u) du] - \int_0^\tau b(u) dW^1(u) - \int_0^\tau c(u) du} \right]. \tag{3.26}$$

Proof. If τ^* is optimal, it minimizes $(E[M(\tau)])^{\frac{1}{1-\rho^2}}$ or equivalently $E[M(\tau)]$ where

$$\begin{aligned} E[M(\tau)] &= E \left[e^{-a(-\delta H(\tau) - \frac{1}{\gamma(1-\rho^2)} \ln E_\tau[l(\tau, T)] + \int_0^\tau I(u) du) - \int_0^\tau b(u) dW^1(u) - \int_0^\tau c(u) du} \right] \\ &= E \left[e^{a(\delta H(\tau) - \int_0^\tau I(u) du) - \int_0^\tau b(u) dW^1(u) - \int_0^\tau c(u) du} E_\tau[l(\tau, T)] \right] \\ &= E \left[e^{a[\delta H(\tau) - H(T) - \int_0^T I(u) du] - \int_0^T b(u) dW^1(u) - \int_0^T c(u) du} \right]. \end{aligned}$$

□

Remark 3. *The expression in (3.26) cannot be evaluated analytically, even with the deterministic model parameters, since it also involves the moment generating function of a lognormal distribution which does not exist. However, in the special case of $\delta(T) = 1$, which means that there is no additional benefit in buying the house at date T , this expression, and hence the value function, can be computed explicitly with the additional assumption of deterministic model parameters. We will call the corresponding value function (respectively, wealth) **reservation utility** (respectively, **wealth**) of the investor when $\tau^* = T$:*

Definition 3. *If the investor does not buy the house until date T so that $\tau^* = T$, then $V = V^T$ is called the reservation utility of the investor, and the corresponding wealth process is called the reservation wealth.*

Corollary 5. *Assume that all the model parameters $\mu, \sigma, \mu^H, \sigma^H, \mu^I$ and σ^I are constant, and $\delta(T) = 1$.*

(a) *Then, for a fixed $\gamma > 0$, the reservation utility is given by*

$$V = -e^{-\gamma(x_0 + \mu^I T - \frac{\mu \rho \sigma^I}{2\sigma} T^2) - \frac{1}{2} \frac{\mu^2}{\sigma^2} T + \frac{1}{6} \gamma^2 (1-\rho^2) (\sigma^I)^2 T^3}. \quad (3.27)$$

(b) *Assume that the value function in (3.27) is the reservation utility ($\tau^* = T$) for all $\rho, -1 < \rho < 1$, and let $f(\rho)$ denote (3.27) as a function of ρ , for $-1 \leq \rho \leq 1$. Then, the maximum is always attained at $\rho = -1$, and the minimum value of f occurs at $\rho_* = \min(1, \frac{3}{2} \frac{\mu}{\gamma \sigma \sigma^I T})$. In particular, when $\frac{3}{2} \frac{\mu}{\gamma \sigma \sigma^I T} \geq 1$, the function $f(\rho)$ is strictly decreasing.*

Proof. (a) To simplify the notation, let $\delta(T) = 1$, and use $W = W^1$. Then, the exponent of the expression in (3.26) with deterministic parameters becomes

$$\begin{aligned} -a \int_0^T I(u)du - \int_0^T b(u)dW(u) - \int_0^T c(u)du &= -a(\mu^I T + \int_0^T \sigma^I W(t)dt) - bW(T) - cT \\ &= -(a\mu^I + c)T - (a\sigma^I T + b)W(T) + a\sigma^I \int_0^T t dW(t) \end{aligned}$$

which has a Gaussian distribution with mean $-(a\mu^I + c)T$ and variance $\frac{1}{3}a^2(\sigma^I)^2 T^3 + ab\sigma^I T^2 + bT^2$. Then, the expectation in (3.26) is equivalent to $E[e^{-(a\mu^I + c)T - (a\sigma^I T + b)W(T) + a\sigma^I \int_0^T t dW(t)}] = e^{-(1-\rho^2)[\gamma\mu^I + \frac{\mu^2}{2\sigma^2}]T + \frac{1}{6}\gamma^2(1-\rho^2)^2(\sigma^I)^2 T^3 + \frac{\gamma(1-\rho^2)\mu\rho\sigma^I}{2\sigma} T^2}$.

By simplifying this expression, raising it to the power $\frac{1}{1-\rho^2}$ and multiplying by $-e^{-\gamma x_0}$, the result follows.

(b) By differentiating the function $f(\rho)$, it is easy to observe that $\frac{3}{2}\frac{\mu}{\gamma\sigma\sigma^I T}$ is the only critical point. The derivative $f'(\rho)$ is negative when $\rho < \frac{3}{2}\frac{\mu}{\gamma\sigma\sigma^I T}$, and positive when $\rho > \frac{3}{2}\frac{\mu}{\gamma\sigma\sigma^I T}$. Moreover, the end points satisfy $f(-1) > f(1)$. So, the result follows from the "first derivative test" on closed intervals. \square

Remark 4. *Although the expression (3.27) is easy to analyze as a function of only one variable, it may be quite complicated when many parameters are analyzed simultaneously, since the strict continuation region ($\tau^* = T$) should be identified for each combination of such parameters.*

Equation (3.26) allows us to easily compute the value function, using standard Monte Carlo simulation methods: Given a time horizon T and some numerical values for the parameters of the model, we consider a grid of points in the interval $[0, T]$ and numerically compute the expectation of (3.25) via (3.26) for each point in the grid; the minimum of (3.26) across all points in the grid is obtained at an approximate optimal stopping time τ^* , and the corresponding expression in (3.26) is an approximation to our value function V . We use the simple expression in (3.26) to study the optimal decision to purchase. That is the objective of the next section.

4 Decision to Purchase

In this section we study the decision to purchase the house and perform comparative statics exercises. Our analysis is based on the numerical tractability of (3.26) that we discussed in the previous section. More explicitly, as we explained in the previous section, given a set of parameter values of the model, it is straightforward to compute numerically the expectation of the right hand side of (3.26) for a given τ . We compute such expectation for a grid of values of τ in the interval $[0, T]$. We would like to characterize the set of parameter values of the model for which the point of the grid corresponding to the minimum value of the expectation in the right hand side of (3.26) is $\tau = 0$. That is, for that set it is optimal to purchase the house immediately. The continuation region is characterized by parameter values for which the value of τ that minimizes the expectation in the right hand side of (3.26) is $\tau > 0$. In order to provide intuition about the problem, we will compute and study *all* the values of τ that minimize the right hand side of (3.26). In an abuse of the terminology, in the rest of this section, we will call the values of τ that minimize the right hand side of (3.26) *optimal investment time* (even if $\tau > 0$, but the only case in which it is the optimal investment time is when $\tau = 0$). Also, we will compute the right hand side of (3.26) for different values of τ and we will call the resulting value *expected terminal utility*.

Remark 5. *Obviously, in our computation of the right hand side of (3.26) we only take into consideration information available at $t = 0$. At every point t we will need to compute the right hand side of (3.26) with the existing information to determine in similar fashion as just described whether it is optimal to exercise or not. What we call “optimal investment time” at $t = 0$ might not be the stopping time when we reach that point (if the stopping time has not arrived before), unless it is $\tau = 0$.*

We want to study the optimal decision to exercise as a function of two parameters, the coefficient of risk aversion γ and the correlation ρ between the house and the stock price processes. We use the following parameter values, $\mu = .11, \sigma = .26, \mu^H = .05, \sigma^H = .11, T = 2.5$ (in years). In addition,

$$\delta(t) = .8 + .08t$$

and

$$I(t) = .35 + .04W^1(t), \quad 0 < t \leq T. \quad (4.1)$$

When we study the optimal purchase decision as a function of the coefficient of risk-aversion γ , we use as value of the correlation coefficient $\rho = 0.1$. When we study the optimal purchase decision as a function of the correlation coefficient ρ we use $\gamma = 2, 4$ and 7 as values of the coefficient of risk-aversion.

Remark 6. *Intuitively, the relevant variable to decide whether the house should be purchased or not is the ratio of the value of the house H to current wealth, X . We denote this ratio by h/x . In the two cases we consider next, we will study values of the ratio h/x at which it is optimal to immediately purchase the house as a function of the coefficient of risk aversion γ and the correlation coefficient ρ , respectively. We denote $(h/x)_*$ the value of the ratio that separates the stopping region (buy the house now) from the continuation region. To provide further intuition, we also compute and plot the value of the ratio for which the value of τ that minimizes the right hand side of (3.26) is T . We denote $(h/x)^*$ the value of the ratio for which $\tau = T$.*

Case 1. *In figure 1, we plot the set of values of γ for which it is optimal to purchase the house as a function of the ratio h/x . $(h/x)_*$, as explained in remark 6, separates the stopping region (values of h/x such that $h/x < (h/x)_*$) from the continuation region (values of h/x such that $h/x > (h/x)_*$). We also plot $(h/x)^*$, described in remark 6. Points between $(h/x)_*$ and $(h/x)^*$ are values of h/X for which the value of τ that minimizes the right hand side of (3.26) is $0 < \tau < T$. As we see, $(h/x)_*$ is decreasing with the degree of risk aversion: everything else equal, a more risk averse investor will buy the house at a lower price. We also observe that for a degree of risk-aversion of about $\gamma = 6.8$ there is a kink and $(h/x)_*$ becomes more convex than $(h/x)^*$. This is due to the shape of the function of equation (3.26). In figure 2, we plot the negative of the right hand side of (3.26) for different values of h/x , as a function of potential purchase times τ when $\gamma = 6$ (expected terminal utility, as described at the beginning of this section). As we can see, it reaches its maximum at $\tau = T$ for high values of h/x , but as we consider lower values of h/x , the maximum jumps to $\tau = 0$. However, in figure 3, for $\gamma = 7.5$, the negative of the expected terminal utility reaches a maximum at $\tau = T$ for high values of h/x ,*

but as it decreases, the maximum is strictly between 0 and T . The value $\gamma = 6.8$ is the point at which the change of shape of the function takes place.

Case 2. We consider now the optimal purchase as a function of the correlation between the price of the house and the stock market. The relationship between the value function V (or the objective function V^τ for a fixed τ) and the correlation coefficient ρ changes for different degrees of risk aversion γ , as illustrated in figure 4. It is clear from the plots that the value function is not a monotone function of ρ , in general. However, it is strictly decreasing on a range between -1 and a positive number, similar to the reservation value function which is strictly decreasing between -1 and ρ_* (see Corollary 5). The value function is maximized at $\rho = -1$, which implies market completeness.

Moreover, the investment boundary is analyzed as a function of ρ in figure 5 (with $\gamma = 7$). The optimal price to purchase the house for a given level of wealth is in general increasing with the absolute value of the correlation coefficient. The graph is not completely symmetric because the expected return on the stock is higher than the expected return on the house, therefore there is a cost in shorting the stock to hedge part of the price of the house.

5 The Case in which a House is Traded for Another

Now, we assume that the investor's problem involves selling his current house at the time of buying a new one where the dynamics of the cash value of the old house satisfies

$$\begin{aligned} dK &= K[\mu^K dt + \sigma^K dW^1] \\ K(0) &= k_0 > 0. \end{aligned}$$

Hence the value matching condition at $t = \tau$ becomes $X(\tau) = X(\tau_-) - \delta H(\tau) + K(\tau)$. Using the notation of the previous sections, we also introduce the following random variable:

Notation 3. For $\tau \in [0, T)$ fixed, define:

$$N(\tau) \triangleq M(\tau)e^{-aK(\tau)} = e^{-a[D(\tau)+K(\tau)+\int_0^\tau I(u)du]-\int_0^\tau b(u)dW^1(u)-\int_0^\tau c(u)du}.$$

Proposition 2. *Assume that all the model parameters are deterministic. Then for a fixed $\tau \in [0, T)$,*

$$V^{\tau, X^\pi(\tau)} = -e^{-\gamma X^\pi(\tau) + \frac{1}{1-\rho^2} \ln E_\tau[l(\tau, T)]}$$

and

$$\begin{aligned} V^\tau &= \sup_{\pi \in \mathcal{U}(0, \tau)} E[V^{\tau, X^\pi(\tau)}] \\ &= e^{-\gamma x_0} (E[N(\tau)])^{\frac{1}{1-\rho^2}}. \end{aligned}$$

Moreover, the optimal portfolio $\hat{\pi}(t)$ is given by

$$\hat{\pi}(t) = \begin{cases} \frac{\mu - \rho\sigma\tilde{\psi}}{\gamma(1-\rho^2)\sigma^2}(t), & 0 \leq t < \tau \\ \frac{\mu - \rho\sigma\phi}{\gamma(1-\rho^2)\sigma^2}(t), & \tau \leq t \leq T \end{cases}$$

where $\tilde{\psi}(t) = -E_t[\mathcal{D}_t N(\tau)/N(t)]$.

Proof. Similar to the ones in the previous section. □

Corollary 6. *For $\gamma > 0$ fixed, the optimal time $\hat{\tau} = \sup_{0 \leq \tau < T} V^\tau$ for the extended problem minimizes*

$$E\left[e^{a(\delta H(\tau) - K(\tau) - \int_0^\tau I(u)du - H(T)) - \int_0^\tau b(u)dW^1(u) - \int_0^\tau c(u)du}\right].$$

Remark 7. *The extension above can be replaced by or combined with a random endowment (e.g. a lump sum amount at the time of retirement) at $t = \tau$, provided that it is adapted to the filtration F^1 only. Again, for more general processes, an optimal portfolio is not guaranteed to exist and we can only get an upper bound for the value function.*

6 Conclusions

In this paper we are interested on the house purchase decision and its interaction with optimal portfolio allocation.

From a financial standpoint, home ownership involves a trade-off: On one hand it provides

some financial advantages, due to savings in rent and/or tax incentives; on the other hand, it locks a sizable proportion of the wealth of the investor in an illiquid security; arguably, even if the investor has perfect access to financial markets and unlimited borrowing possibilities, it is not possible to fully hedge the house price risk which will be, at best, partially -but not fully- correlated with financial markets.

We consider the problem of a risk-averse investor who maximizes expected utility from final wealth and faces that trade-off. Final wealth here represents the value of the total savings of agents at a time -like retirement- at which they might want to start using up those savings for the rest of their lives -which we do not model here. For CARA utility, we compute the optimal investment strategy before and after the purchase decision, and then characterize the optimal stopping time. Our results allow us to perform numerical exercises to study the effect of different parameter values on the optimal acquisition policy, as well as on the optimal investment strategy. For example, the higher the risk-aversion the earlier (that is, the lower price at which) the investor buys the house. We also study the effect of the correlation between stocks and the house price on the purchase decision.

References

- [1] Brendle S. and R. Carmona (2004). Hedging in partially observable markets. Preprint.
- [2] Cauley, S. D., A. D. Pavlov and E. S. Schwartz (2005). Homeownership as a constraint on asset allocation. *Journal of Real Estate Finance and Economics*, 34, 283-311.
- [3] Cocco, J. F. (2005). Portfolio choice in the presence of housing. *Review of Financial Studies*, 18 (2), 535-567.
- [4] Cocco, J. F., F. J. Gomez and P.J. Maenhout (2005). Consumption and portfolio choice over the life-cycle. *Review of Financial Studies*, 18 (2), 491-533.
- [5] Cvitanić J. and I. Karatzas (1992). Convex duality in constrained portfolio optimization. *Annals of Applied Probability* 2, 767-718.
- [6] Flavin, M. and T. Yamashita (2002). Owner-occupied housing and the composition of the household portfolio over the life cycle. *American Economic Review*, 92, 345-362.
- [7] Grossman, S.J. and G. Laroque (1990). Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. *Econometrica*, 58, 25-51.
- [8] Hugonnier and Morellec (2007) Corporate control and real investment in incomplete markets, *Journal of Economic Dynamics and Control*, 31 (5), 1781-1800.
- [9] Karatzas I. and S. E Shreve (1998). *Methods of Mathematical Finance*, Springer-Verlag
- [10] Karatzas I. and Wang (2001). Utility maximization with discretionary stopping. *SIAM Journal on Control & Optimization* 39, 306-329
- [11] Miao, J. and N. Wang (2007). Investment, consumption, and hedging under incomplete markets. *Journal of Financial Economics* 86, 608-642.
- [12] Piazzesi, M., M. Schneider and S. Tuzel (2007). Housing, consumption and asset pricing. *Journal of Financial Economics* 83, 531-570.
- [13] Tebaldi, C. and E. Schwartz (2007). Illiquid assets and optimal portfolio choice. Working paper, UCLA.

- [14] Viceira, L. (2001). Optimal portfolio choices for long-horizon investors with nontradable labor income. *Journal of Finance*, 56 (2), 433-470.
- [15] Yao, R. and H. H. Zhang (2005). Optimal consumption and portfolio choices with risky housing and borrowing constraint. *Review of Financial Studies*, 18 (1), 197-239.

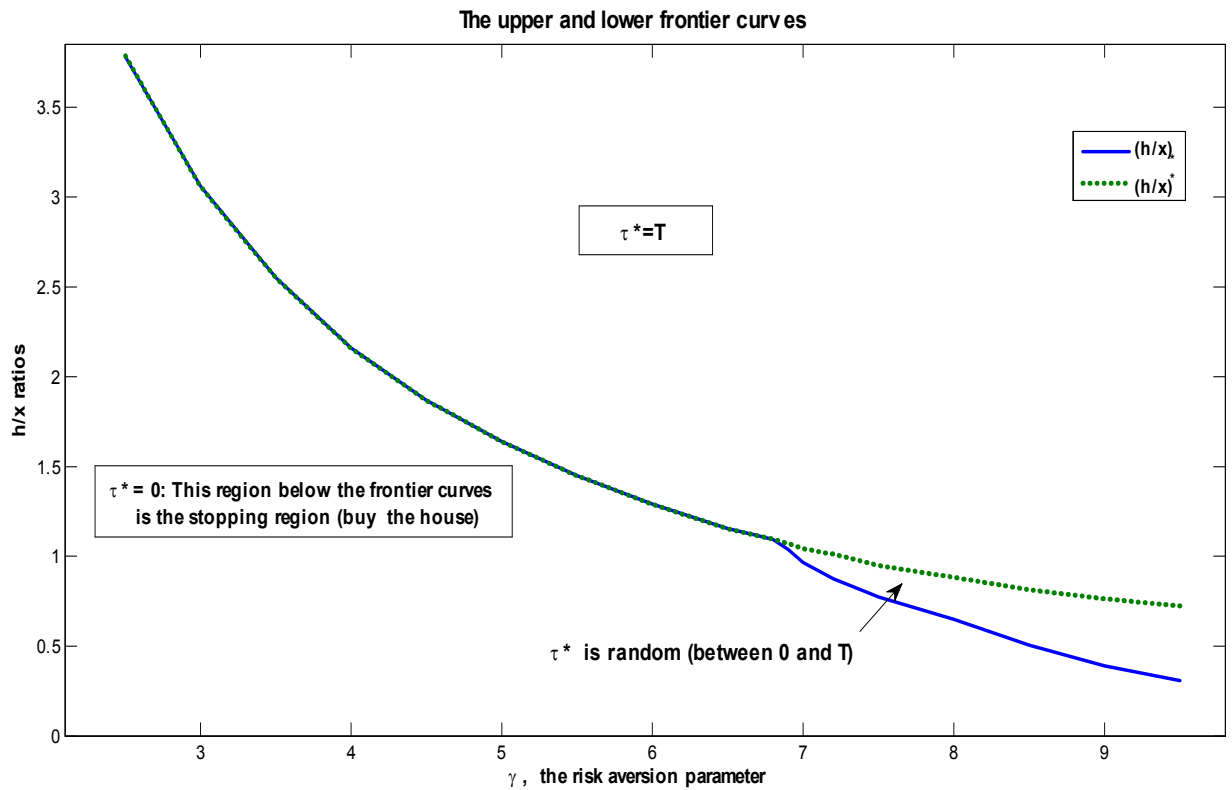


Figure 1: For a certain range of relatively lower risk aversion values (e.g. when $\gamma < 6.8$), the optimal real estate investment time shifts from $\tau^* = 0$ to $\tau^* = T$ instantly at the boundary h/x values where two frontier curves coincide. At higher risk aversion levels, this transition becomes rather gradual.

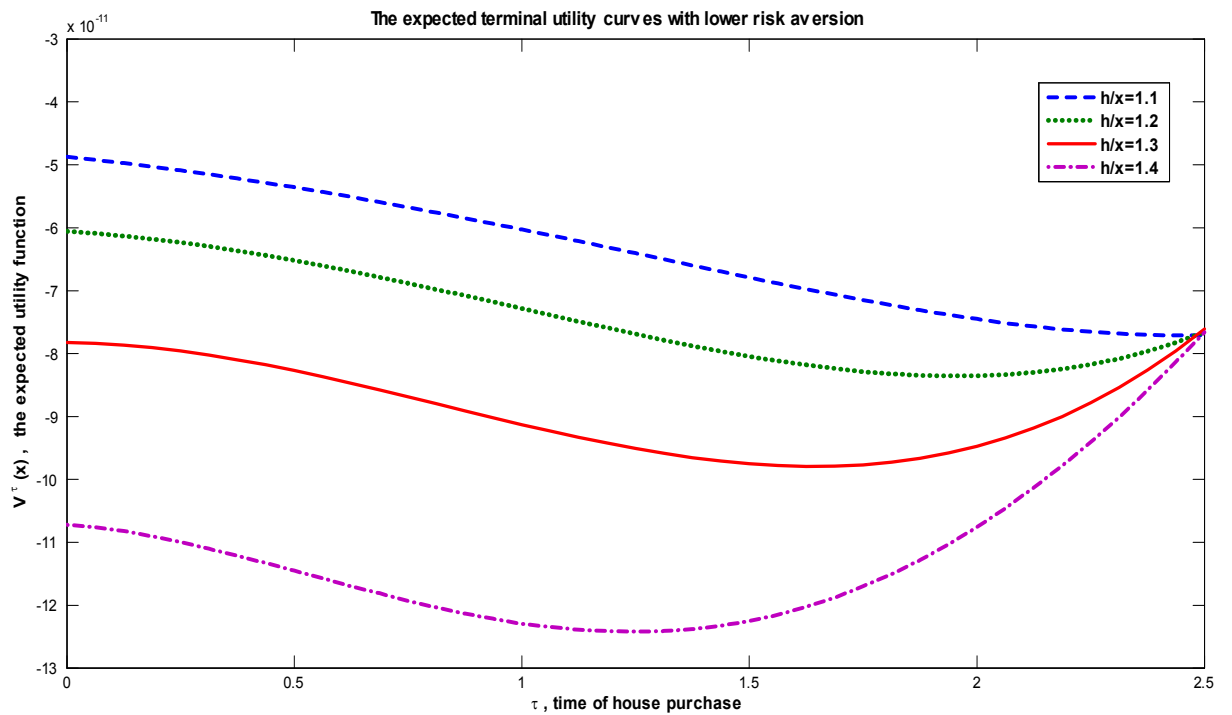


Figure 2: When the risk aversion parameter γ is below 6.8, this figure is a typical graph of the expected utility of the investor's total terminal wealth. The optimal investment time occurs either at $\tau^* = 0$ or at $\tau^* = 2.5$ (for relatively higher h/x ratios). The risk aversion parameter $\gamma = 6$ is used to plot the figure.

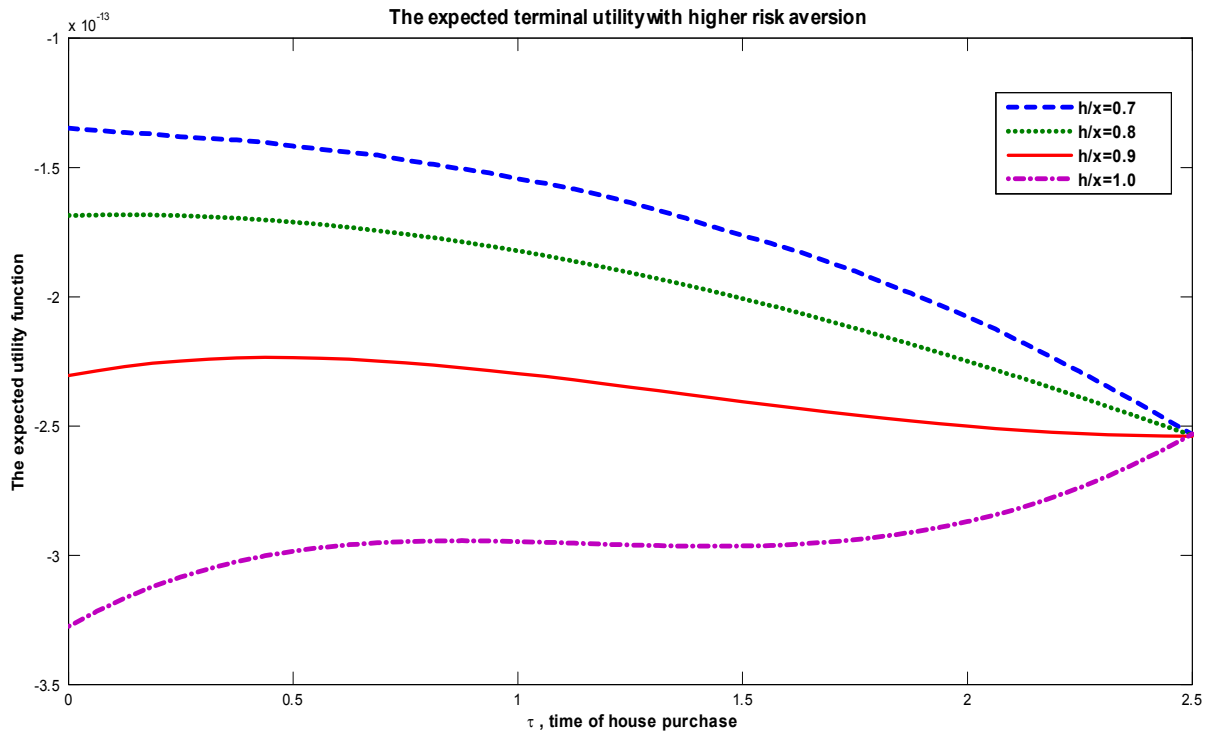


Figure 3: When the risk aversion is relatively high, e.g. if γ is above 6.8, this is a typical graph of the expected terminal utility of wealth (the parameter value $\gamma = 7.5$ is used here). The optimal investment time is $\tau^* = 0$ for low h/x ratios (e.g. when $h/x < 0.8$); it is between 0 and T for relatively moderate h/x ratios (e.g. for 0.8, 0.9 above); and finally $\tau^* = 2.5$ for higher h/x ratios (e.g. h/x is 1 or larger).

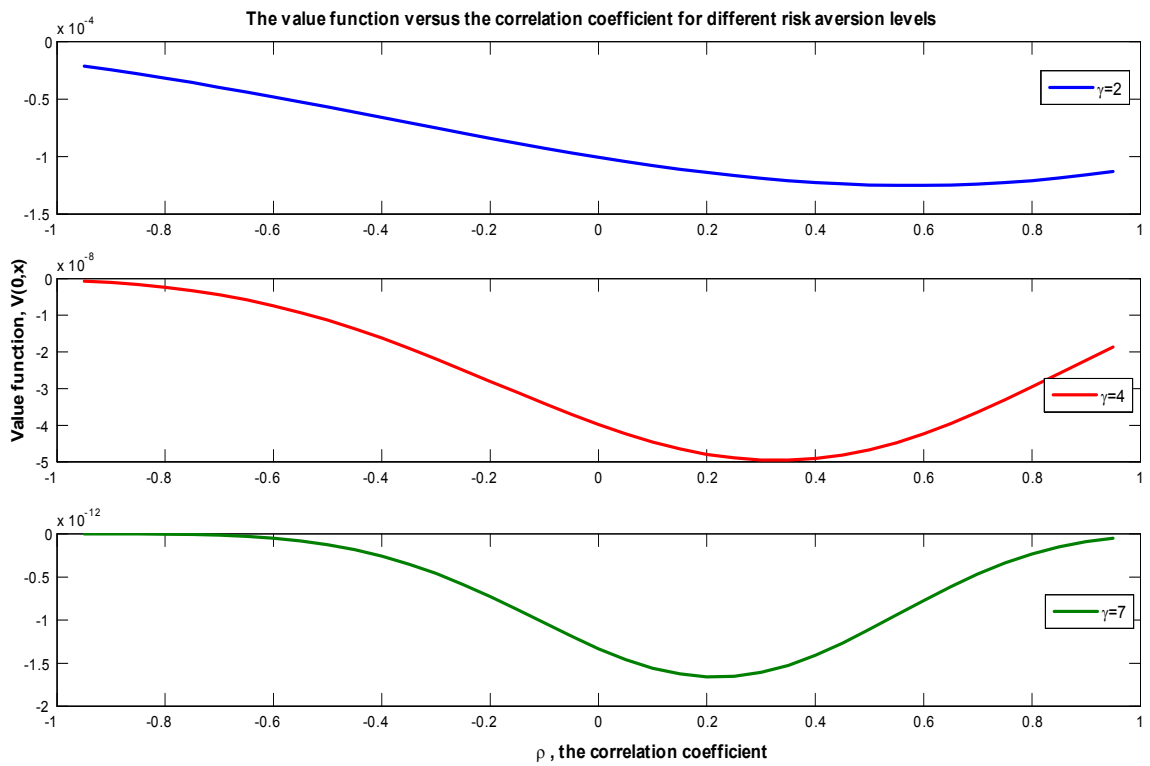


Figure 4: All the value functions are computed at the h/x ratio 1 and at $\tau = 0$ which is the optimal time to buy the house (for $\gamma = 7$ in the third panel, the optimal time is slightly positive but the corresponding difference is negligible). The value function is not monotone in ρ and is larger when the correlation is negative.

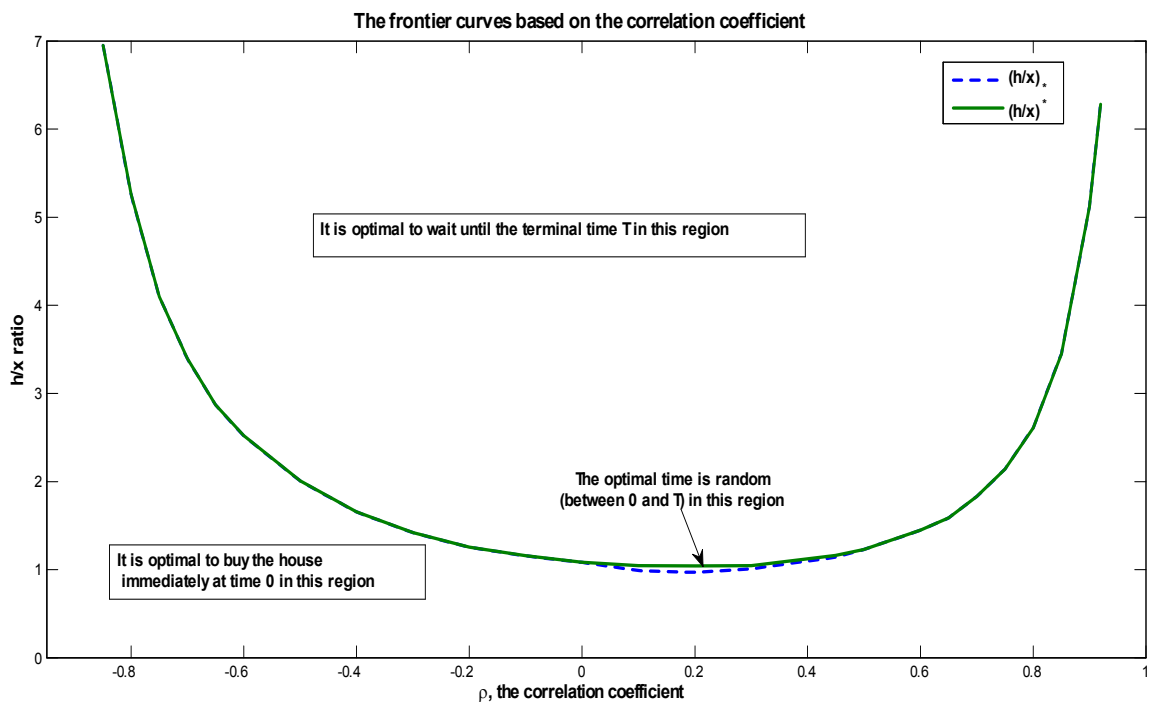


Figure 5: The boundary h/x ratios form concave frontier curves as functions of the correlation coefficient ρ . The upper and the lower curves coincide except on a range of low positive ρ values.